TROPICAL GEOMETRY

GRIGORY MIKHALKIN

This Arbeitstagung 2009 talk surveys the current state of development of Tropical Geometry. We do not make an attempt to make an exhausting survey, but rather choose some particular topics to make it a collection of some short stories from the area.

1. Introduction and a (basic) example

Recall that as Mathematics operates with rather abstract notions, many notions may admit several different-looking (and perhaps still sufficiently abstract) *realizations*.

For example, let us consider (algebro-geometric) curves. These are 1-dimensional algebraic varieties. Their classical realization (XIX century) is provided by Riemann surfaces, i.e. smooth 2-dimensional manifolds with a choice of complex structure in their tangent bundle. The story generalizes to higher-dimensional algebraic varieties, but it is especially easy is dimension 1. In this dimension the complex structure is given by an endomorphism J in every tangent space with the property that $J^2 = -1$ (i.e. an almost complex structure). Furthermore, a complex structure on a Riemann surface may be described by a metric of constant curvature. Projective curves correspond to compact surfaces. The genus of a curve is one half of its first Betti number (i.e. the number of cycles). It can also be computed as the dimension of the space of holomorphic 1-forms on the surface.

Compact tropical curves can be realized as so-called metric graphs (considered up to an equivalence). These are finite graphs where the interior of each edge is enhanced with an inner metric. We impose the requirement that the length of an edge adjacent to a 1-valent vertex must be infinite. Such an edge is called a *leaf edge*. The genus of a tropical curve is the number of cycles. It can be also computed as the dimension of the space of tropical 1-forms on the graph.

To get tropical curves we consider metric graphs equivalent if one can obtained from the other by contracting a leaf edge. Clearly genus depends only on an equivalence class. Note that all genus 0 curves are equivalent. Thus the tropical rational curve is unique just as in the classical case. Curves of positive genus $g > 0$ admit unique minimal

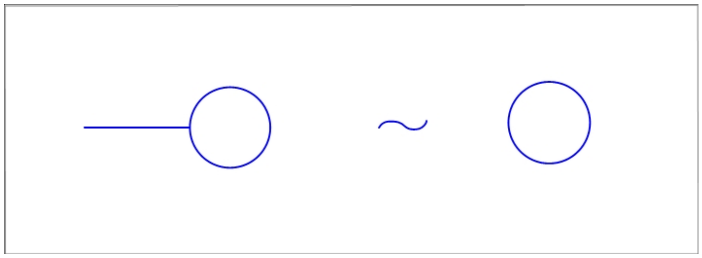

FIGURE 1. Equivalent elliptic curves

models – graphs without leaves. Generically such graphs are 3-valent and have $3g-3$ edges of some length. Thus the space of all curves of genus $g>1$ is $(3g-3)$-dimensional.

It is easy to see that tropical curves possess many other properties that we can expect from projective curves. In particular, any curve of genus smaller than 3 admits a hyperelliptic involution. In the same time a generic genus 3 curve is not hyperellptic, but trigonat, etc. To a tropical curve we may associate its Picard group, its jacobian varieties. Many classical 19th century theorems about Riemann surfaces (such as Abel-Jacobi, Riemann-Roch, the Riemann theorem on Θ-divisor, etc) admit straightforward and easy-to-visualize tropical counterparts, cf. [9].

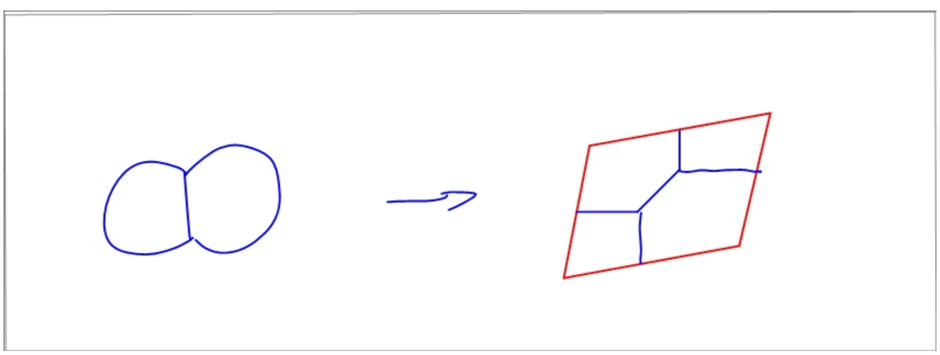

FIGURE 2. A genus 2 curve as a Θ-divisor in its Jacobian variety

2. Tropical varieties and morphisms, the balancing condition

As a set tropical numbers $\mathbb{T}$ coincide with the half-open real line $[-\infty,+\infty)$. There are two tropical arithmetic operations (which we denote in quotation marks to distinguish them from standard arithmetic

operations): tropical addition "$x+y$" $=\max\{x,y\}$ and tropical multiplication "xy" $=x+y$. Clearly we get tropical division "x/y" $=x-y$. However there is no chance for tropical subtraction as tropical addition is idempotent: $x+x=x$. Actually in most geometric constructions we can easily avoid using arithmetics at all.

Let us consider the affine n-space $\mathbb{T}^n$ and the n-torus $(\mathbb{T}^\times)^n=\mathbb{R}^n$. Here $\mathbb{T}^\times=\mathbb{T}\smallsetminus\{0_\mathbb{T}\}=\mathbb{R}$ as the neutral element under addition is $0_\mathbb{T}=-\infty$. Tropical structure in these spaces is given by the sheaf of tropical regular functions that are obtained from tropical rational functions by restricting them to open sets where they are convex. The geometric structure that encodes such a sheaf is the integer-affine structure on $\mathbb{R}^n$. Thus tropical varieties can be thought as polyhedral complexes enhanced with an integer-affine structure.

There are local and global conditions on such an enriched polyhedral complex $(X,\mathcal{O})$. Locally we require that $(X,\mathcal{O})$ is equivalent to $(\mathbb{T}^n,\mathcal{O}_{\mathbb{T}^n})$. Equivalence here is generated by smooth divisors, i.e. those that are themselves smooth $(n-1)$-dimensional tropical varieties. Globally we require a certain finite type condition. The resulting object is a (smooth) tropical manifold. Tropical manifolds come with (equivalent) local embeddings to $\mathbb{T}^N$, $N\geq n$, that exhibit them as piecewise-linear polyhedral complexes $Q\subset\mathbb{R}^N$ (or, rather their closures in $\mathbb{T}^N\supset\mathbb{R}^N$). By a piecewise-linear polyhedral complex we mean a union of convex polyhedra in $\mathbb{R}^N$. Furthermore, we require that the slope of each face E is integer, i.e. the vector subspace $V_E\subset\mathbb{R}^N$ parallel to E is generated by integer vectors.

Any local model polyhedron complex $Q\subset\mathbb{R}^n$ is *balanced*. This is a property at $(n-1)$-dimensional faces of Q. Let $E\subset Q$ be an $(n-1)$-face and $F_1,\dots,F_k$ be the n-facets adjacent to Q. Each F_j defines a vector v_j in the quotient vector space $\mathbb{R}^{N-n}=\mathbb{R}^N/V_E$,namely a primitive integer vector parallel to the image of F_j in the projection. The balancing condition is formulated as

$$\sum_j v_j=0\in\mathbb{R}^N/V_E.$$

It is always satisfied if Q is locally equivalent to $\mathbb{T}^n$. Furthermore we have some additional (finer) properties at faces of codimension greater than 1.

Alternatively, we may define a class of tropical n-spaces where we only impose the balancing condition at the faces of codimension 1 and no additional conditions at higher codimensions. Furthermore, at the n-faces we may put integer weights. These are the so-called tropical cycles. A cycle is *effective* if the weights are positive. We may define

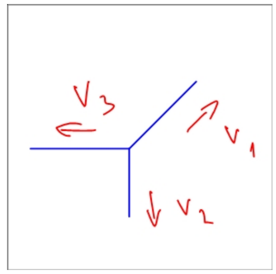

FIGURE 3. Balancing condition

positive *multiplicities* at the points of such cycles. If all these multiplicities equal to one then the cycle is called a homological tropical variety (or a pseudomanifold). Such spaces are locally given by matroids and their local realizability by complex effective cycles depends on the realizability of the corresponding matroid, cf. [7].

All morphisms between homological varieties are given by integer affine-linear maps of the ambient varieties. Morphisms between smooth tropical manifolds are more restricted, they are given by regular functions. E.g. scaling by 2 of all the edges is induced by an integer affine-linear map of the ambient $\mathbb{R}^2$, but is not an endomorphism of a tripod graph (as a smooth tropical 1-manifold). Note that the number of critical points of this would-be endomorphism is negative and thus it is never approximated by a complex map.

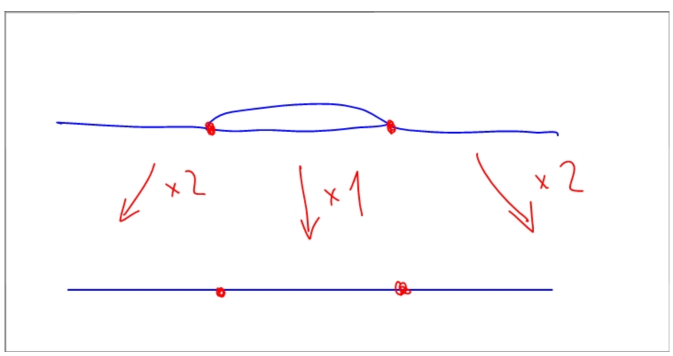

FIGURE 4. A (realizable) degree 2 map from an elliptic curve to $\mathbb{TP}^1$.

3. Interactions between tropical and classical worlds

Connection between complex and tropical numbers is provided by $\log_t : \mathbb{C} \to \mathbb{T},\ z \mapsto \log_t |z|$. When $t \to \infty$ the map becomes more and

more homomorphism-like. Images of complex affine varieties under the map

$$\mathrm{Log}_t : \mathbb{C}^n \to \mathbb{T}^n$$

obtained by coordinate-wise application of $\log_t$ are called *amoebas* and carry significant information about geometry of complex varieties. Even better picture is obtained after consideration of images of families $V_t \subset \mathbb{C}^n$ under Log_t when $t \to \infty$. The limits of these images are (perhaps singular) tropical varieties.

More generally, tropical varieties X sometimes can be obtained as a result of collapse $\lambda_t : \mathcal{X}_t \to X$ of families of complex varieties $\mathcal{X}_t$. Such a collapse is easy to produce in the case when X is a tropical curve (with the help of decomposition into pairs-of-pants) or if X is a smooth tropical complete intersection (by tropicalizing the defining equations). Tropical varieties may be enhanced with *phases* responsible for gluing data. The phase-tropical structure can also be included in the approximation data.

For curves the phase data amount to the twist for gluing pairs-of-pants. If the curve is given by a 3-valent graph and we fix a cyclic orientation for the edges adjacent to every 3-valent vertex we have a canonical (untwisted) choice of gluing. E.g. if we have a plane curve $h : C \to \mathbb{TP}^2$ the cyclic order is given by the ambient plane. The untwisted phase-tropical curves give the so-called simple Harnack curves, cf. [5].

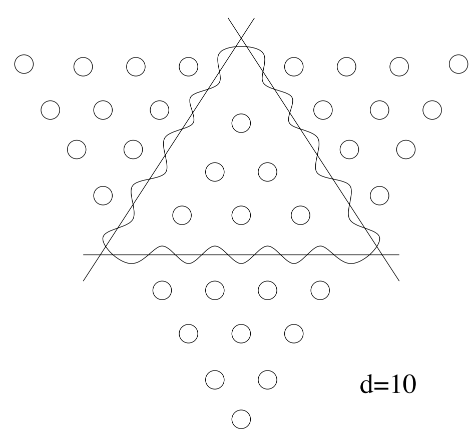

Figure 5. A Harnack curve of degree 10.

Suppose that $h: C \to X$ is a tropical morphism, where C is a curve and X is a complete intersection. We may approximate C by a complex family $\mathcal{C}_t$ and X with a complex family $\mathcal{X}_t$. But can we approximate h with a family of holomorphic maps $H_t : \mathcal{C}_t \to \mathcal{X}_t$. It turns out that it is not always so. Nevertheless the following theorem provides a criterion for such realizability.

It can be shown (with the help of the tropical Riemann-Roch theorem) that any tropical curve $h: C \to X$ in X has a deformation space of dimension at least $-K_X.[h(C)] + (1-g)(\dim X - 3)$.

Definition 3.1. A tropical map $h: C \to X$ is called *regular* if the dimension of the deformation space of h is $-K_X.[h(C)]+(1-g)(\dim X-3)$. Otherwise h is called *superabundant*.

Theorem 1 ([6]). *A regular tropical morphism $h: C \to X$ is approximable by a family of holomorphic maps $H_t : \mathcal{C}_t \to \mathcal{X}_t$.*

There are many examples of non-realizable superabundant curves. For example a map $h: C \to \mathbb{TP}^1$ from an elliptic curve depicted on Figure 6 is realizable only if the lengths a and b are equal. This is a special case of a realizability of genus 1 curves found by David Speyer [10].

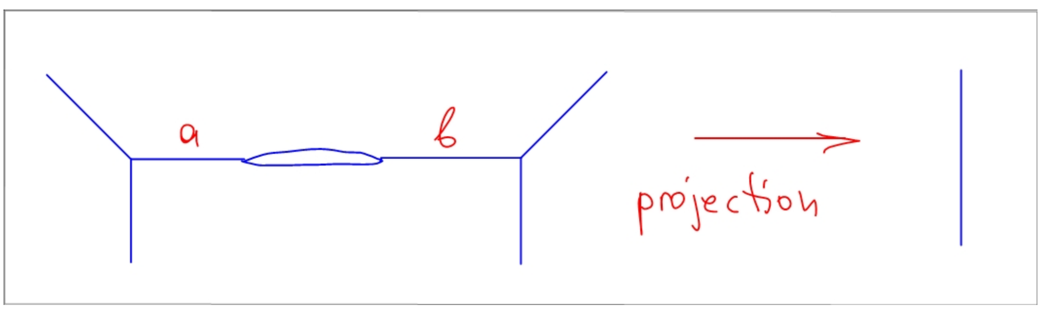

FIGURE 6. A non-realizable superabundant map from an elliptic curve to $\mathbb{TP}^1$.

4. APPLICATIONS TO COMPLEX AND REAL ENUMERATIVE GEOMETRY

Theorem 1 allows to replace certain (regular) enumerative problems in classical (complex and real) geometry with the corresponding tropical problems. Often the latter problems are much more manageable combinatorially.

For example, consider the problem of finding the number of complex (or real) curves of degree d and genus g passing through $3d - 1 +$

g generic points in $\mathbb{P}^2$ or $2d$ points in $\mathbb{P}^3$. In the case of the real enumerative problems the curves have to be counted with signes defined by Welschinger [12], [13] in the case of genus 0 (in the case of positive genus we do not consider the real case at all as at the moment there is no corresponding real invariant defined).

Theorem 1 may be used to reduce both complex and real problem to enumeration of tropical curve passing through the corresponding collection of points in $\mathbb{TP}^2$ or $\mathbb{TP}^3$. Each such tropical curve acquires a multiplicity that might be different for the instances of real and complex enumeration.

In the corresponding tropical enumerative problem we may choose the points to be well stretched vertically. Tropical curves passing through such points are described by the so-called *floor diagrams*, see [1]. Every floor diagram (with marking) encodes a unique tropical curve. Without the marking the floor diagram is an even better-looking combinatorial object. As it was shown in [2] in the planar genus 0 case it corresponds to a tree on d vertices, so there is d^{d-2} of them. Thus the number of corresponding complex and real curve (the genus 0 Gromov-Witten and Welschinger numbers for $\mathbb{P}^2$) can be interpreted as two (different) statistics on trees. Both of this statistics are non-negative and coincide on trees corresponding to floor diagrams where the weight of all edges are equal to 1 (otherwise they differ by scaling depending on these weights). In particular, this implies the results of Itenberg-Kharlamov-Shustin [4] on logarithmic asymptotics of the Welschinger invariants.

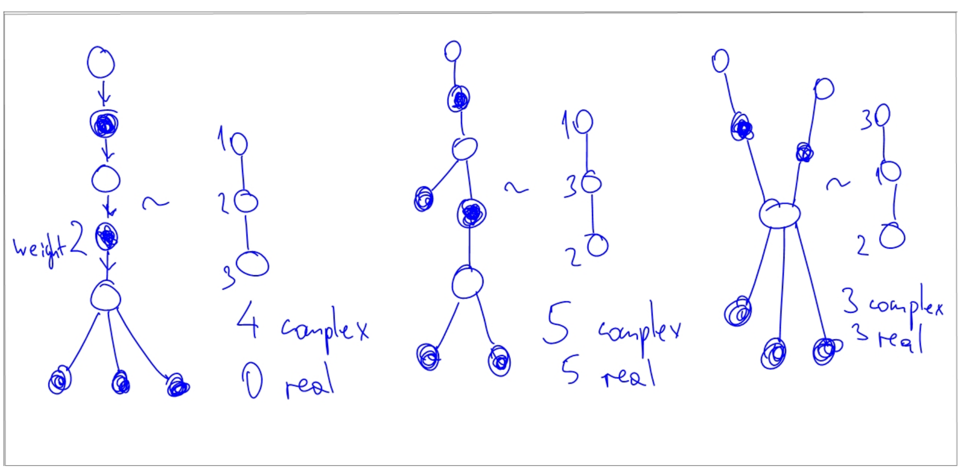

FIGURE 7. Floor diagrams computing the number of complex and real rational cubic curves through 8 generic points in $\mathbb{P}^2$.

5. Patchworking of real varieties

Yet another direction of applications of tropical geometry is based on interpretation of Theorem 1 as a generalization of Viro's patchworking theorem [11]. Recall that the Viro theorem allows to find real curves embedded to the plane with controlled topology in the context of the first part of Hilbert's 16th problem. Theorem 1 allows to generalize this construction to immersed curves in the plane as well as to algebraic knots and links in $\mathbb{RP}^3$.

To illustrate what happens with the analogue of Hilbert's question in dimension 3 (particularly in the positive genus case) we list a classification of smooth curves of degree 5 and genus 1 in $\mathbb{RP}^3$ recently obtained by Mikhalkin and Orevkov [8]. All topological types in this case are depicted on Figure 8.

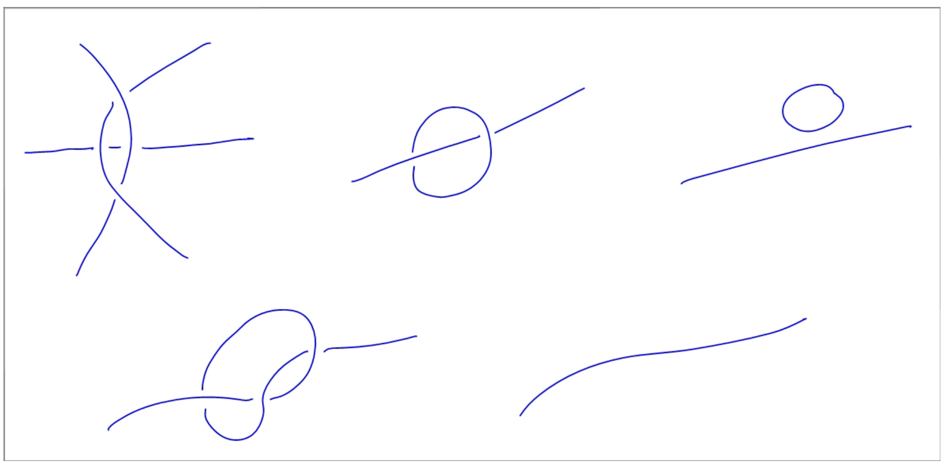

Figure 8. Topological types of degree 5 genus 1 knots in $\mathbb{RP}^3$.

As it was shown by Harnack [3] the number of components of a real curve of genus g can not exceed $g+1$. The following theorem comes as an application of Theorem 1 and allows to represent any projective link in $\mathbb{RP}^3$ by an algebraic curve of the minimal possible genus (without specifying the degree).

Theorem 2. *Let $L \subset \mathbb{RP}^3$ be a link in $g+1$ components (i.e. a smoothly embedded disjoint union of $g+1$ circles). There exists a smooth algebraic curve of genus g isotopic to L.*

Clearly this theorem provides a generalization for the well-known theorem that any knot can be approximated by a rational curve. Finding the minimal degree of an algebraic realization for most simple knots and links in $\mathbb{RP}^3$ is a challenging question.

I would like to finish this talk with the question on the knot type of rational curves passing through $2d$ points in $\mathbb{RP}^3$. A rational curve of odd degree in $\mathbb{RP}^3$ is homologous to $[\mathbb{RP}^1] \in H_1(\mathbb{RP}^3)$. We say that it is *knotted* if it is not isotopic to $\mathbb{RP}^1 \subset \mathbb{RP}^3$.

Question 1. *Suppose that d is a large odd degree. Is it true that for any generic collection of $2d$ points in $\mathbb{RP}^3$ there exists a knotted rational curve passing through the points. Are there any knot types that are forced to appear in such enumeration?*

In this question we restrict to the case odd degree as 3-dimensional Welschinger invariant is non-trivial then. (An easy symmetry consid-erarion shows that it is zero if d is even.) Perhaps a similar question is also sensible for the even degree.

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