Quasi-conformal geometry and word hyperbolic Coxeter groups

Marc Bourdon (joint work with Bruce Kleiner)

Arbeitstagung, 11 june 2009

In [6] J. Heinonen and P. Koskela develop the theory of (analytic) modulus in metric spaces, and introduce the notion of Loewner space. They establish that many results concerning the classical quasi-conformal geometry on Euclidean spaces are valid for on Loewner spaces. In geometric group theory the regularity properties of quasi-symmetric homeomorphisms between Loewner spaces are responsible for several rigidity phenomena generalizing Mostow's rigidity. Otherwise only few examples of Loewner spaces are known, these include the boundaries of rank one symmetric spaces, the boundaries of some fuchsian buildings, and some exotic self-similar spaces.

Cannon's conjecture states that every word hyperbolic group whose boundary is homeomorphic to the 2-sphere acts by isometries properly discontinuously and cocompactly on the real hyperbolic space $\mathbb{H}^3$. It can be seen as a group theoretical analogue of Thurton's hyperbolization conjecture recently solved by G. Perelman. As a tool to approach Cannon's conjecture, various notions of combinatorial modulus have been developed by several authors (*e.g.* [3], [4], [1], [5])).

This talk will focus on the combinatorial modulus. It reports on a recent joint work with B. Kleiner [2]. A combinatorial version of the Loewner property, called the *combinatorial Loewner property*, is presented. It is weaker than Heinonen-Koskela's, indeed if Z is a Q-Loewner space then every metric space quasi-symmetrically homeomorphic to Z satisfies the combinatorial Q-Loewner property. We suspect that in most of the interesting cases – like the boundaries of word hyperbolic groups – a converse is also true, namely that if a metric space admits the combinatorial Q-Loewner property then it is quasi-symmetrically homeomorphic to a Q-Loewner space.

Our main results concern the combinatorial modulus on boundaries of word hyperbolic Coxeter groups. We obtain a sufficient condition for such a boundary to satisfy the combinatorial Loewner property, and use this to exhibit a number of examples, some old and some new. As an application of our techniques we obtain a proof of Cannon's conjecture in the special case of Coxeter groups. This result was essentially known. Our view is that the principal value of the proof is that it illustrates the feasibility of the asymptotic approach (using the ideal boundary and modulus estimates), and it may suggest ideas for attacking the general case. It gives also a new proof of the Andreev's theorem about the Coxeter hyperbolic polytopes in $\mathbb{H}^3$, in the case when the prescribed dihedral angles are submultiples of π.

References

[1] M. Bonk, B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres,* Invent. Math., 150 (2002), 1247–1287.

[2] M. Bourdon, B. Kleiner, *Combinatorial modulus, combinatorial Loewner property and Coxeter groups,* Preprint (2009).

[3] J. W. Cannon, *The combinatorial Riemann mapping theorem,* Acta Math. **173** (1994), no. 2, 155–234.

[4] J. W. Cannon, W.J. Floyd, W.R. Parry, *Sufficiently rich families of planar rings,* Ann. Acad. Sci. Fenn. Math. **24** (1999), no. 2, 265–304.

[5] P. Haïssinsky, *Empilements de cercles et modules combinatoires,* Annales de l'Institut Fourier, to appear.

[6] J. Heinonen, P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry,* Acta Math. **181** (1998), 1-61.

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