# Discontinuous Groups on pseudo-Riemannian Spaces

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Toshiyuki Kobayashi

(the University of Tokyo)

http://www.ms.u-tokyo.ac.jp/˜toshi/



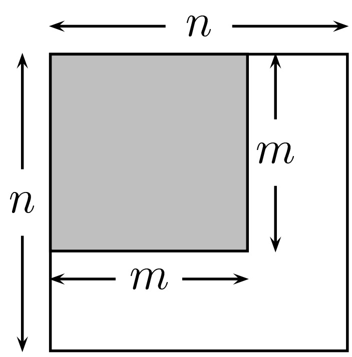

# Compact quotients $\Gamma\backslash SL(n)/SL(m)$

Problem (Existence problem for uniform lattice):

Does there exist compact Hausdorff quotients of

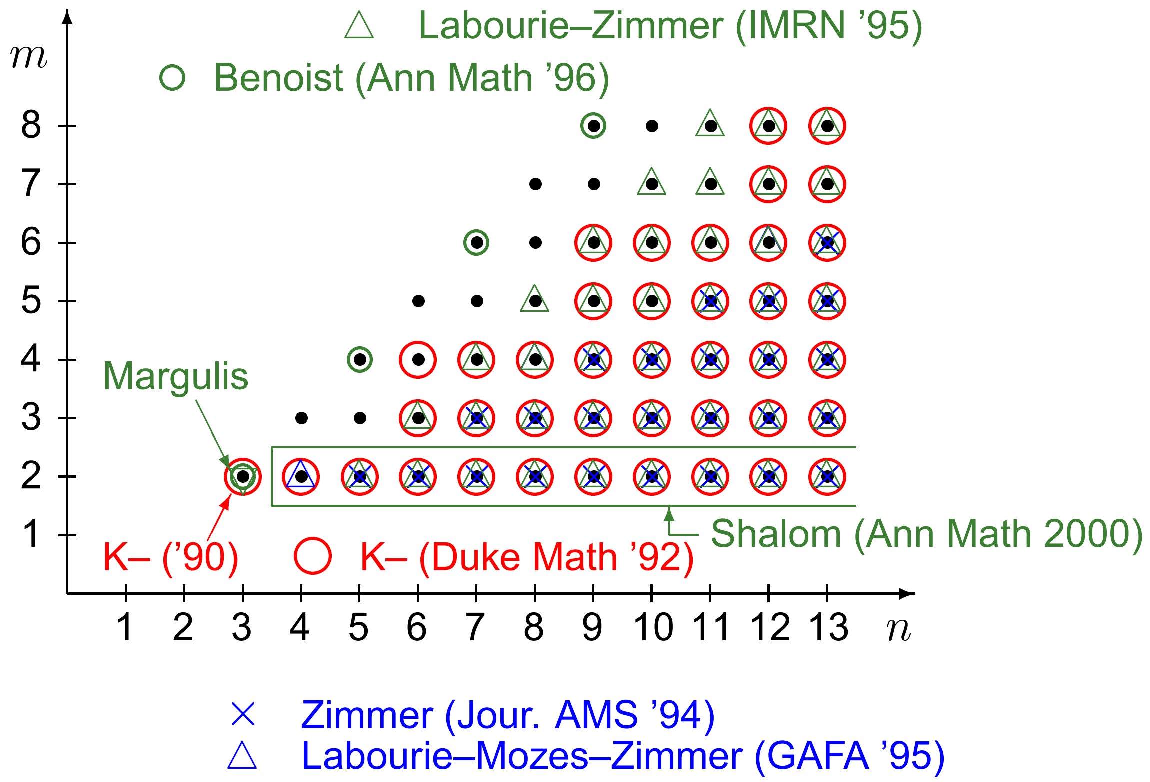
$$SL(n,\mathbb{F})/SL(m,\mathbb{F}) \quad (n > m,\ \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$$

by discrete subgps $\Gamma$ of $SL(n,\mathbb{F})$?



# Compact quotients for $SL(n)/SL(m)$

Uniform lattice does not exist for the following $(n, m)$:






# $SL(n)/SL(m)$ case

> Conjecture  For any $n > m > 1$, there does not exist uniform lattice for $SL(n)/SL(m)$.

Affirmative results:

| | | |
|---|---|---|
| K– | criterion of proper actions | $\frac{n}{3} > [\frac{m+1}{2}]$ |
| Zimmer | orbit closure thm (Ratner) | $n > 2m$ |
| Labourier–Mozes–Zimmer | | |
| | ergodic action | $n \geq 2m$ |
| Benoist | criterion of proper actions | $n = m + 1,\ m$ even |
| Margulis | unitary representation | $(n \geq 5, m = 2)$ |
| Shalom | unitary representation | $n \geq 4, m = 2$ |

# Non-Riemannian homo. spaces

Discrete subgp $\overset{\not\Rightarrow}{\Leftarrow}$ Discontinuous gp

for non-Riemannian homo. spaces

General Problem

How does a local geometric structure affect the global nature of manifolds?

New phenomena & methods?



# 2. Complex symmetric structure

$G/K$: Riemannian symmetric space

$\Downarrow$ complexification

$G_{\mathbb{C}}/K_{\mathbb{C}}$: complex symmetric space

Fact (Borel 1963) Compact quotients exist for $^\forall$Riemannian symm sp. $G/K$.

Conj. Compact quotients exist for $G_{\mathbb{C}}/K_{\mathbb{C}}$
$\iff G_{\mathbb{C}}/K_{\mathbb{C}} \approx S^7_{\mathbb{C}}$ or complex group mfd

$\Leftarrow$ proved by K–Yoshino 05,

$\Rightarrow$ remaining case $S^{4k-1}_{\mathbb{C}}$, $k \geq 3$ (Benoist, K– )

# Space forms (examples)

Space form $\cdots$ $\begin{cases} \text{Signature } (p,q) \text{ of pseudo-Riemannian metric } g \\ \text{Curvature } \kappa \in \{+,0,-\} \end{cases}$

E.g. $q = 0$ (Riemannian mfd)

| sphere $S^n$ | $\mathbb{R}^n$ | hyperbolic sp |
|---|---|---|
| $\kappa > 0$ | $\kappa = 0$ | $\kappa < 0$ |

E.g. $q = 1$ (Lorentz mfd)

| de Sitter sp | Minkowski sp | anti-de Sitter sp |
|---|---|---|
| $\kappa > 0$ | $\kappa = 0$ | $\kappa < 0$ |



# Space form problem

Space form problem for pseudo-Riemannian mfds

**Local Assumption**

signature $(p,q)$, curvature $\kappa \in \{+,0,-\}$

$\Downarrow$

**Global Results**

- Do compact forms exist?
- What groups can arise as their fundamental groups?

# Compact space forms

$(p, q)$: signature of metric, curvature $\kappa \in \{+, 0, -\}$

Assume $p \geq q$ (without loss of generality).

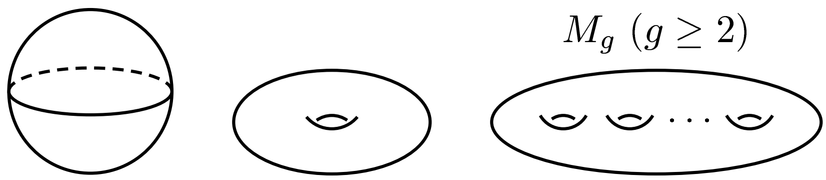
- $\kappa > 0$: Calabi–Markus phenomenon
  (Calabi, Markus, Wolf, Wallach, Kulkarni, K–)
- $\kappa = 0$: Auslander conjecture
  (Bieberbach, Auslander, Milnor, Margulis, Goldman, Abels, Soifer, ...)
- $\kappa < 0$: Existence problem of compact forms



# 2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2, 0)$)

$M_g$ $(g \geq 2)$



curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Lorentz case ($\Longleftrightarrow$ signature $(1, 1)$)
compact forms do NOT exist
for $\kappa > 0$ and $\kappa < 0$

# Compact space forms ($\kappa < 0$)

- Riemannian case $\cdots$ hyperbolic space

Compact quotients

$\iff$ Cocompact discont. gp for $O(n,1)/O(n) \times O(1)$

$\iff$ Cocompact discrete subgp of $O(n,1)$

(uniform lattice)

Exist by $\underbrace{\text{Siegel, Borel–Harish-Chandra, Mostow–Tamagawa}}_{\text{arithmetic}}$,

$\underbrace{\text{Vinberg, Gromov–Piateski-Shapiro}}_{\text{non-arithmetic}} \cdots$



# Existence of compact forms

- For pseudo-Riemannian mfd of signature $(p, q)$

~~Thm~~ Conjecture Compact space forms of $\kappa < 0$ exist

$\Longleftarrow$ $\Longrightarrow$ ① $q$ any, $p = 0$ ($\leftrightarrow \kappa > 0$)

② $q = 0$, $p$ any (hyperbolic sp)

③ $q = 1$, $p \equiv 0 \mod 2$

④ $q = 3$, $p \equiv 0 \mod 4$ (pseudo-Riemannian)

⑤ $q = 7$, $p = 8$

$\Longleftarrow$ True (Proved (1950–2005))

(①② (Riemmanian) ; ③④⑤ (pseudo-Riemannian) Kulkarni, K– )

$\Longrightarrow$ Partial answers:

$q = 1$, $p \leq q$, or $pq$ is odd

Hirzebruch's proportionality principle (K–Ono)

# Infinitesimal approximation

$$G = K \exp \mathfrak{p} \implies G_\theta = K \ltimes \mathfrak{p} \quad \text{(Cartan motion gp)}$$

$$\boxed{G/H = O(p, q+1)/O(p,q) \implies G_\theta/H_\theta}$$

<u>Thm</u> (K–Yoshino, 2005)
Compact forms of $G_\theta/H_\theta$ exist $\iff p \equiv 0 \mod 2^{\varphi(q)}$

Here, $\varphi(q) = \left[\frac{q}{2}\right] + \begin{cases} 0 & (q \equiv 0, 6, 7 \mod 8) \\ 1 & (q \equiv 1, 2, 3, 4, 5 \mod 8) \end{cases}$

| <u>E.g.</u> | | | |
|---|---|---|---|
| | $q = 0$ | | $p$ any |
| | $q = 1$ | $\varphi(1) = 1$ | $p \equiv 0 \mod 2$ |
| | $q = 3$ | $\varphi(3) = 2$ | $p \equiv 0 \mod 4$ |
| | $q = 7$ | $\varphi(7) = 3$ | $p \equiv 0 \mod 8$ |



# Radon–Hurwitz number (1922)

<u>Def.</u> (Radon–Hurwitz number)
$\rho(p) := 8\alpha + 2^\beta$
if $p = u \cdot 2^{4\alpha+\beta}$ ($u$: odd, $0 \leq \beta \leq 3$)

$$p \equiv 0 \mod 2^{\varphi(q)} \iff q < \rho(p)$$

Radon–Hurwitz number (1922)

$\Downarrow$

Adams: vector fields on sphere (1962)

$\Downarrow$

Uniform lattice for $G_\theta/H_\theta$ (2005)

# General idea: Compact-like actions

Non-compact Lie groups

occasionally behave nicely
when embedded in $\infty$-dim groups
as if they were

compact groups
(very nice behaviours)



# Compact-like linear/non-linear actions

$L \curvearrowright \mathcal{H}$ (linear)

Unitarizability
$=$ existence of $L$-invariant inner product

Discrete decomposability
$=$ no continuous spectrum
in the $L$-irreducible decomposition

$L \curvearrowright M$ (non-linear)

Proper acions/properly discontinuous actions
$=$ The action map $\begin{matrix} L \times M \to M \times M \\ (g, x) \mapsto (x, g \cdot x) \end{matrix}$ is proper.

# Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space, unitary reprn.
$L \curvearrowright \mathcal{H}$ discrete decomposability

$\cdots$ $L$ behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

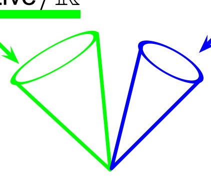
$M$: topological space
$L \curvearrowright M$ proper actions

$\cdots$ $L$ behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group



# Criterion of admissible restriction

Theorem A (Criterion) (K– Ann Math '98, Progr Math '05)
Let $G' \underset{\text{reductive}/\mathbb{R}}{\subset} G$ and $\pi \in \widehat{G}$. If



$(\star)$ $\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$ in $\underset{\mathbb{R}^n}{\overset{\|}{\sqrt{-1}\mathfrak{t}^*}}$,

$\Longleftrightarrow$ $\pi|_{K'}$ is $K'$-admissible.

In particular, the restriction $\pi|_{G'}$ is $G'$-admissible.
(discretely decomposable & of finite multiplicities)

Proof uses micro-local analysis.

# $\pitchfork$ and $\sim$ (definition)

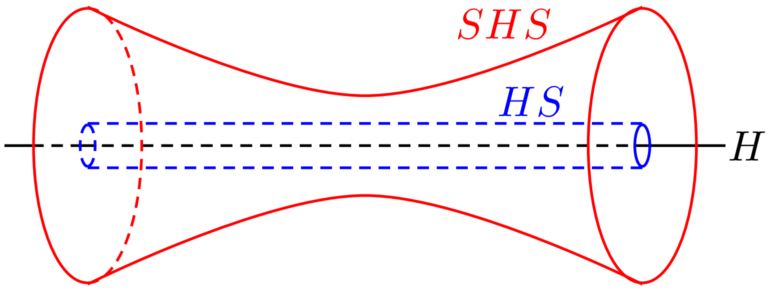
$$L \quad \subset \quad G \quad \supset \quad H$$

Idea: forget even that $L$ and $H$ are group

Def. (K– )

1) $L \pitchfork H \iff \overline{L \cap SHS}$ is compact
for $^{\forall}$ compact $S \subset G$

2) $L \sim H \iff {}^{\exists}$ compact $S \subset G$
s.t. $L \subset SHS$ and $H \subset SLS$.






# $\pitchfork$ and $\sim$

$$L \quad \subset \quad G \quad \supset \quad H$$

Forget even that $L$ and $H$ are group

1) $L \pitchfork H \iff$ generalization of proper actions
2) $L \sim H \iff$ economy in considering

Meaning of $\pitchfork$:

$L \pitchfork H \iff L \curvearrowright G/H$ proper action

for closed subgroups $L$ and $H$

$\sim$ provides economies in considering $\pitchfork$

$$H \sim H' \implies \boxed{H \pitchfork L \iff H' \pitchfork L}$$

# Criterion for $\pitchfork$ and $\sim$

$G$: real reductive Lie group
$G = K\exp(\mathfrak{a})K$: Cartan decomposition
$\nu$: $G \to \mathfrak{a}$: Cartan projection (up to Weyl gp.)

<u>Thm B</u> (K– , Benoist)
1) $L \sim H$ in $G$ $\iff$ $\nu(L) \sim \nu(H)$ in $\mathfrak{a}$.
2) $L \pitchfork H$ in $G$ $\iff$ $\nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$.

abelian

Special cases include

(1)'s $\Rightarrow$ : <u>Uniform</u> bounds on errors in eigenvalues when a matrix is perturbed.

(2)'s $\Leftrightarrow$ : Criterion for <u>properly discont. actions</u>.



# Criterion for compact-like actions

$G$ : reductive Lie group $\supset K$
$\cup$ $\cup$
$G'$ : reductive subgp $\supset K'$
$\mu : T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*$ momentum map

<u>Thm A</u> $\pi \in \widehat{G}$, $G' \subset G$
$\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$
$\Longrightarrow \pi|_{G'}$ is discrete decomposable.

$G$ : reductive Lie gp, $G \supset L, H$ (subsets)
$\nu : G \to \mathfrak{a}$ (Cartan projection)

<u>Thm B</u> (proper action)
$L \pitchfork H$ in $G \iff \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$

# Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space
$L \curvearrowright \mathcal{H}$ discrete decomposability
$\cdots$ $L$ behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

$M$: topological space
$L \curvearrowright M$ proper actions
$\cdots$ $L$ behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group



# Local $\Longrightarrow$ Global

$G \supset H$ reductive Lie groups
$\Longrightarrow$ $G/H$ pseudo-Riemannian homo. sp

Cor (Criterion for the Calabi–Markus phenomenon)
Any discont. gp for $G/H$ is finite
$\Longleftrightarrow \mathrm{rank}_{\mathbb{R}}\, G = \mathrm{rank}_{\mathbb{R}}\, H$

Application (space form of signature $(p, q)$, $\kappa < 0$)
Exists a space form $M$ s.t. $|\pi_1(M)| = \infty$
$\Longleftrightarrow p > q$ or $(p, q) = (1, 1)$
(Calabi, Markus, Wolf, Kulkarni, Wallach)
$p > q + 1 \Longrightarrow {}^{\exists} M$ with free non-commutative $\pi_1(M)$

# Rigidity, stability, and deformation

$$\begin{tikzcd}[row sep=small] \Gamma \arrow[rrd, bend left, "\text{properly discont.}"] & & \\ \cap & G \curvearrowright & X \\ \Gamma' \arrow[rru, dashed, bend right] \arrow[u, phantom, "\subset" description] & & \end{tikzcd}$$

Suppose $\Gamma'$ is 'close to' $\Gamma$

| | |
|---|---|
| (R) (local rigidity) | $\Gamma' = g\Gamma g^{-1}$ $({}^{\exists}g \in G)$ |
| (S) (stability) | $\Gamma' \curvearrowright X$ properly discont. |

In general,

- (R) $\Rightarrow$ (S).
- (S) may fail (so does (R)).



# Local rigidity and deformation

$\Gamma \subset G \curvearrowright X = G/H$ cocompact, discontinuous gp

<u>General Problem</u>

1. When does local rigidity (R) fail?
2. Does stability (S) still hold?

<u>Point</u>: for non-compact $H$

1. (good aspect) There may be large room for deformation of $\Gamma$ in $G$.
2. (bad aspect) Properly discontinuity may fail under deformation.

# Rigidity Theorem

① $\Gamma \curvearrowright G/\{e\} \iff (\Gamma \times 1) \curvearrowright (G \times G)/\Delta G$ ②

$\Gamma \subset G$ simple Lie gp

Fact (Selberg–Weil's local rigidigy, 1964)
$^\exists$uniform lattice $\Gamma$ admitting continuous deformations for ①
$\iff G \approx SL(2, \mathbb{R})$ (loc. isom).

Thm (K– )
$^\exists$uniform lattice $\Gamma$ admitting continuous deformations for ②
$\iff G \approx SO(n+1, 1)$ or $SU(n, 1)$ $(n = 1, 2, 3, ...)$.

Local rigidity (R) may fail. Stability (S) still holds.

$\cdots$ Solution to Goldman's stability conjecture (1985), 3-dim case



# Compact-like linear/non-linear actions

$\mathcal{H} = L^2(G/H), L^2(G/\Gamma)$: Hilbert space
$L \curvearrowright \mathcal{H}$ discrete decomposability
$\cdots$ $L$ behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

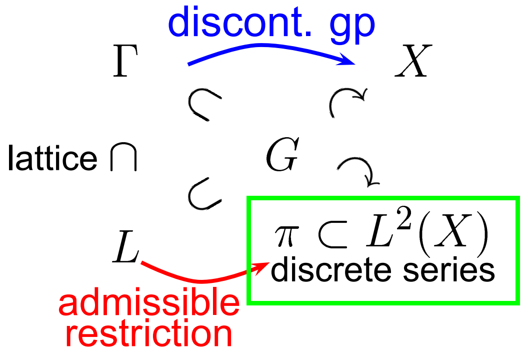
$M = G/H$: topological space
$L \curvearrowright M$ proper actions
$\cdots$ $L$ behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group

# Interacting example

$$(G, L, H) = (SO(4,2),\ SO(4,1),\ U(2,1))$$

Tessellation of pseudo-Riemannian mfd $X$

$X = SO(4,2)/U(2,1) \quad (\underset{\text{open}}{\subset} \mathbb{P}^3\mathbb{C})$




$\pi$: discrete series of $G$ with GK-dim 5

(quarternionic discrete series)

$\Longrightarrow \pi|_L$ is $L$-admissible



# References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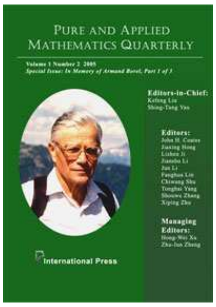


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For more references:
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