Implosion in symplectic, hyperkähler and algebraic geometry

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SYMPLECTIC REDUCTION

(X, ω) symplectic manifold

ω is a closed nondegenerate 2-form on X; locally $X \cong \mathbb{R}^{2m}$ with

$$\omega = \sum_{1 \leq j \leq m} dx_j \wedge dx_{m+j}$$

Example: $X = T^*M$ cotangent bundle

K compact Lie group with Lie algebra $\mathfrak{k}$ acting on (X, ω)

$\mu : X \to \mathfrak{k}^*$ **moment(um) map** satisfies

$$d\mu_x(\xi).a = \omega_x(\xi, a_x) \qquad \forall x \in X, \xi \in T_xX, a \in \mathfrak{k}$$

and μ is K-equivariant (for the coadjoint action on $\mathfrak{k}^*$).

Special case: (X, ω) is **Kähler** and K acts holomorphically; then the action extends to

$$G = K_{\mathbb{C}} = \text{complexification of } K$$

[e.g. $SL(n; \mathbb{C})$ is the complexification of $SU(n)$].

Let $\zeta \in \mathfrak{k}^*$ be a regular value of $\mu : X \to \mathfrak{k}^*$.
Let K_ζ be its stabiliser for the coadjoint action.
Then the

Marsden-Weinstein reduction at ζ

$$\mu^{-1}(\zeta)/K_\zeta$$

is a symplectic orbifold.

Often we take $\zeta = 0$:

$$X//K = \mu^{-1}(0)/K$$

'symplectic quotient'

Kähler case: $\mu^{-1}(0)/K =$ (open subset of X)$/G$ inherits Kähler structure.

N.B. grad$\mu(x).a = i\, a_x \quad \forall a \in \mathfrak{k}$

$X//K = \mu^{-1}(0)/K$ has symplectic/Kähler structure with more serious singularities when 0 is not a regular value of μ.

Example:

$$X = (\mathbb{P}^1)^4$$

where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = S^2 \subseteq \mathbb{R}^3$.

$K = SU(2)$ acting on X via rotations of S^2

$G = K_{\mathbb{C}} = SL(2;\mathbb{C})$ Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}$$

moment map $\mu : X \to \mathfrak{k}^* \cong \mathbb{R}^3$ given by

$$\mu(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4.$$

$\mu^{-1}(0)/K$ represented by 'balanced' configurations of four points on S^2.

Hyperkähler quotients

X hyperkähler manifold, complex structures i, j, k with $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$ etc, metric g, Kähler forms ω_1, ω_2, ω_2

compact group K acting on X preserving i, j, k, g

Hyperkähler moment map

$$\mu = (\mu_1, \mu_2, \mu_3) : X \to \mathfrak{k}^* \otimes \mathbb{R}^*$$

Often fix the complex structure i and write $\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : X \to \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbb{C}}$ with $\mu_{\mathbb{R}} = \mu_1$ and $\mu_{\mathbb{C}} = \mu_2 + i\mu_3$; then $\mu_{\mathbb{C}}$ is holomorphic wrt i.

Examples: $\mathbb{H}^n$, $T^* K_{\mathbb{C}}$
(closures of) coadjoint orbits in $\mathfrak{k}^*_{\mathbb{C}}$
(Kronheimer, Kovalev, Nakajima, Kobak–S ...)

Hyperkähler quotient

$$X///K = \mu^{-1}(0)/K = \mu_{\mathbb{C}}^{-1}(0)//K$$

(Hitchin, Karlhede, Lindström, Rocek)

Mumford's geometric invariant theory (GIT)

G complex reductive linear algebraic group
X complex projective variety acted on by G

We require a **linearisation** of the action (i.e. an ample line bundle L on X and a lift of the action to L; think of $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho : G \to GL(n+1)$).

$$
\begin{array}{ccccl}
X & \Rightarrow & A(X) & = & \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X \\
| & & & & = \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\
| & & \cup\!| & & \\
\downarrow & & & & \\
X//G & \Leftarrow & A(X)^G & & \text{algebra of invariants}
\end{array}
$$

G reductive implies that $A(X)^G$ is a finitely generated graded complex algebra so that $X//G = \mathrm{Proj}(A(X)^G)$ is a projective variety.

The rational map $X - \to X//G$ fits in a diagram

$$\begin{array}{rccccl} & X & --\to & X//G & \text{cx proj variety} \\ & \cup & & \| & \\ \text{semistable} & X^{ss} & \xrightarrow{\text{onto}} & X//G & \\ & \cup & & \cup & \text{open} \\ \text{stable} & X^{s} & \longrightarrow & X^{s}/G & \end{array}$$

where the morphism $X^{ss} \to X//G$ is G-invariant and surjective.

Topologically $\boxed{X//G = X^{ss}/\sim}$ where

$$x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset.$$

G reductive $\Leftrightarrow$ G is the complexification $K_{\mathbb{C}}$ of a maximal compact subgroup K (for example $SL(n) = SU(n)_{\mathbb{C}}$), and then

$$x \in X^{ss} \Leftrightarrow \overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$$

for a suitable moment map μ for the K-action, and

$$X//G = \mu^{-1}(0)/K = X//K$$

NB There is a slight conflict of notation here.

$\boxed{\text{What can we do if } G \text{ is not reductive?}}$

Problem: We can't define a projective variety

$$X//G = \mathrm{Proj}(A(X)^G)$$

as $A(X)^G$ is not necessarily finitely generated, so can we still define a sensible 'quotient' $X//G$?

Theorem (Doran–K,...): Let G be a linear algebraic group over $\mathbb{C}$ acting linearly on $X \subseteq \mathbb{P}^n$. Then X has open subsets X^s ('**stable** points') and X^{ss} ('**semistable** points'), a **geometric quotient** $X^s \to X^s/G$ and an '**enveloping quotient**' $X^{ss} \to X//G$. Moreover *if* $A(X)^G$ is finitely generated then $X//G = \mathrm{Proj}(A(X)^G)$.

$$\begin{array}{rccccl} & X & --\rightarrow & X//G & \\ & \cup & & || & \\ \text{semistable} & X^{ss} & \longrightarrow & X//G & \\ & \cup & & \cup & \text{open} \\ \text{stable} & X^s & \longrightarrow & X^s/G & \end{array}$$

Warning: $X//G$ is **not necessarily projective** and $X^{ss} \to X//G$ is **not necessarily onto**.

Simple example: $\mathbb{C}^+$ acting on $\mathbb{P}^n$
We can choose coordinates in which the generator of $Lie(\mathbb{C}^+)$ has Jordan normal form with blocks of size $k_1+1,\ldots,k_q+1$. The linear $\mathbb{C}^+$ action therefore extends to $G = SL(2)$ with

$$\mathbb{C}^+ = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}\} \leqslant G$$

via $\mathbb{C}^{n+1} \cong \bigoplus_{i=1}^q Sym^{k_i}(\mathbb{C}^2)$.

In fact in this case the invariants are finitely generated (Weitzenbock) so we can define

$$\mathbb{P}^n//\mathbb{C}^+ = \mathrm{Proj}((\mathbb{C}[x_0,\ldots,x_n])^{\mathbb{C}^+}).$$

N.B. Via $(g,x) \mapsto (g\mathbb{C}^+, gx)$ we have

$$G \times_{\mathbb{C}^+} \mathbb{P}^n \cong (G/\mathbb{C}^+) \times \mathbb{P}^n \cong (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}^n$$

$$\subseteq \mathbb{C}^2 \times \mathbb{P}^n \subseteq \mathbb{P}^2 \times \mathbb{P}^n$$

and so

$$\boxed{\mathbb{P}^n//\mathbb{C}^+ \cong (\mathbb{P}^2 \times \mathbb{P}^n)//SL(2)}$$

$$
\begin{array}{ccc}
\mathbb{P}^2 \times \mathbb{P}^n & - - \rightarrow & \mathbb{P}^2 \times \mathbb{P}^n /\!/ G \\
\cup & & || \\
\mathbb{P}^n = \{[1:0:1]\} \times \mathbb{P}^n & - - \rightarrow & \mathbb{P}^n /\!/ \mathbb{C}^+ \\
\cup & & || \\
(\mathbb{P}^n)^{ss} & \stackrel{\text{not nec onto}}{\longrightarrow} & \mathbb{P}^n /\!/ \mathbb{C}^+ \\
\cup & & \cup \\
(\mathbb{P}^n)^s & \longrightarrow & (\mathbb{P}^n)^s / \mathbb{C}^+
\end{array}
$$

Example when $(\mathbb{P}^n)^{ss} \to \mathbb{P}^n /\!/ \mathbb{C}^+$ is *not* onto:

$\mathbb{P}^3 = \mathbb{P}(Sym^3(\mathbb{C}^2)) = \{$ 3 unordered points on $\mathbb{P}^1\}$.

Then $(\mathbb{P}^3)^{ss} = (\mathbb{P}^3)^s$ is
$\{$ 3 unordered points on $\mathbb{P}^1$, at most one at $\infty\}$

and its image in

$$\mathbb{P}^3 /\!/ \mathbb{C}^+ = (\mathbb{P}^3)^s / \mathbb{C}^+ \ \sqcup\ \mathbb{P}^3 /\!/ SL(2)$$

is the open subset $(\mathbb{P}^3)^s / \mathbb{C}^+$ which does not include the 'boundary' points coming from

$$0 \in \mathbb{C}^2 \subseteq \mathbb{P}^2.$$

SYMPLECTIC IMPLOSION (Guillemin, Jeffrey, Sjamaar 2001)

Ingredients: (X, ω) symplectic manifold

Hamiltonian action of compact connected group K

$\mu : X \to \mathfrak{k}^*$ moment map

T maximal torus of K, Lie algebra $\mathfrak{t} \subseteq \mathfrak{k}$

Weyl group $W = N_T/T$ acts on $\mathfrak{t}$ and $\mathfrak{t}^*$ which decompose into Weyl chambers.

$\mathfrak{t}^*_+ =$ positive Weyl chamber $\cong \mathfrak{t}^*/W \cong \mathfrak{k}^*/K$.

Recall $K_\zeta = \{k \in K | (Ad^* k)\zeta = \zeta\}$. Its commutator subgroup $[K_\zeta, K_\zeta]$ is generated by the commutators $khk^{-1}h^{-1}$ for $k, h \in K_\zeta$.

The **imploded cross-section** of X is

$$X_{impl} = \mu^{-1}(\mathfrak{t}^*_+)/\sim$$

where $x \sim y \Leftrightarrow x = ky$ for some $k \in [K_\zeta, K_\zeta]$ with

$$\zeta = \mu(x) = \mu(y) \in \mathfrak{t}^*_+.$$

Examples: (1) $K = SU(2)$.

$$\mathfrak{t}^*_+ = [0, \infty) = \{0\} \sqcup (0, \infty)$$

$$X_{impl} = \frac{\mu^{-1}(0)}{SU(2)} \sqcup \mu^{-1}((0, \infty))$$

(2) $K = SU(3)$.

Over the interior points of $\mathfrak{t}^*_+$ no collapsing occurs since $[K_\zeta, K_\zeta] = [T, T]$ is trivial.
Over nonzero boundary points of $\mathfrak{t}^*_+$ we have $K_\zeta \cong U(2)$ and $[K_\zeta, K_\zeta] \cong SU(2)$.
Over $0 \in \mathfrak{t}^*_+$ we have $K_\zeta = SU(3) = [K_\zeta, K_\zeta]$.

X_{impl} inherits a symplectic structure and T-action with moment map $X_{impl} \to \mathfrak{t}^*_+ \subseteq \mathfrak{t}^*$ induced by the restriction of μ.

K acts on itself by left translation and hence on $T^*K \cong \mathfrak{k}^* \times K$ with moment map

$$\mu(p,q).a = p \cdot a_q \quad \forall a \in \mathfrak{k},\, q \in K,\, p \in T_q^*K = \mathfrak{k}^*.$$

$(T^*K)_{impl}$ '**universal imploded cross-section**' is an affine algebraic variety over $\mathbb{C}$.

In general

$$X_{impl} \cong (X \times (T^*K)_{impl})//K$$

which is an algebraic variety if X is algebraic.

Example: $K = SU(2) \cong S^3 \subseteq \mathbb{C}^2$

$$(T^*SU(2))_{impl} = \frac{\mu^{-1}(0)}{SU(2)} \sqcup \mu^{-1}((0,\infty))$$

$$\cong \{\text{point}\} \sqcup (\mathbb{C}^2 \setminus \{0\}) \cong \mathbb{C}^2$$

with induced T-action multiplication by t^{-1}.

Link with Kähler/algebraic geometry:
$G = K_{\mathbb{C}}$ complexification of K;
B Borel subgroup of G (maximal soluble subgp) such that $G = KB$ and $K \cap B = T$.
$N \subseteq B$ maximal unipotent subgroup of G;
$B = T_{\mathbb{C}} N$ with $T_{\mathbb{C}}$ complex torus.

FACT: $K_{\mathbb{C}}/N$ is a quasi-affine variety whose algebra of regular functions $\mathcal{O}(K_{\mathbb{C}}/N) = \mathcal{O}(K_{\mathbb{C}})^N$ is finitely generated, so that $K_{\mathbb{C}}/N$ has a canonical affine completion

$$K_{\mathbb{C}}//N = Spec(\mathcal{O}(K_{\mathbb{C}})^N).$$

Thm (GJS): $K_{\mathbb{C}}//N$ has a K-invariant Kähler structure such that it is symplectically iso to the universal imploded cross-section $(T^*K)_{impl}$.

Cor: X projective variety acted on by $K_{\mathbb{C}} \Rightarrow$

$$X_{impl} \cong (X \times (K_{\mathbb{C}}//N))//K_{\mathbb{C}} \cong X//N.$$

Generalised symplectic implosion:
$G = K_{\mathbb{C}}$ complexification of K;
P parabolic subgroup of G;
U_P unipotent radical of P.

FACT: G/U_P is a quasi-affine variety whose algebra of regular functions $\mathcal{O}(G/U_P) = \mathcal{O}(G)^{U_P}$ is finitely generated, so that G/U_P has a canonical affine completion

$$G//U_P = Spec(\mathcal{O}(G)^{U_P}).$$

Thm: $G//U_P$ has a K-invariant Kähler structure such that it can be described symplectically as a generalised universal imploded cross-section $(T^*K)^{(P)}_{impl}$.

Cor: X proj variety acted on linearly by $G \Rightarrow$ its U_P-invariants are finitely generated and

$$X//U_P \cong (X \times (G//U_P))//G \cong X^{(P)}_{impl}.$$

Towards hyperkähler implosion

Recall: X hyperkähler manifold, complex structures i, j, k, metric g, Kähler forms ω_1, ω_2, ω_3

compact group K acting on X preserving i, j, k, g

Hyperkähler moment map

$$\mu = (\mu_1, \mu_2, \mu_3) : X \to \mathfrak{k}^* \otimes \mathbb{R}^3$$

Hyperkähler quotient $X///K = \mu^{-1}(0)/K$

Hyperkähler implosion X_{hkimpl} should be stratified hyperkähler with an induced T-action.

Look for the **universal hyperkähler implosion**

$$(T^*K_{\mathbb{C}})_{\text{hkimpl}}$$

with an induced hyperkähler action of $T \times K$ and then define

$$X_{\text{hkimpl}} = (X \times (T^*K_{\mathbb{C}})_{\text{hkimpl}})///K.$$

Recall the **symplectic case**:

$$(1) \quad (T^*K)_{\text{impl}} = (K \times \mathfrak{t}^*_+)/\sim$$

with $(k,\zeta) \sim (k',\zeta')$ iff $\zeta = \zeta'$, $k'k^{-1} \in [K_\zeta, K_\zeta]$, so $(T^*K)_{\text{impl}}//_\zeta T = K$-coadjoint orbit of ζ.

$$(2) \qquad (T^*K)_{\text{impl}} = K_{\mathbb{C}}//N$$

where $K_{\mathbb{C}} \cong T^*K \cong K \times \mathfrak{k}^*$.

The universal hyperkähler implosion $(T^*K_{\mathbb{C}})_{\text{hkimpl}}$ should be the complex symplectic quotient of $T^*K_{\mathbb{C}} \cong K_{\mathbb{C}} \times \mathfrak{k}^*_{\mathbb{C}}$ by N

$$K_{\mathbb{C}} \times (\mathfrak{k}_{\mathbb{C}}/\text{Lie}N)^*//N = (K_{\mathbb{C}} \times \mathfrak{n}^0)//N.$$

Exists (with finitely generated invariants)?
Hyperkähler? Geometry?

Consider $K = SU(n)$. Co-adjoint orbits of $K_{\mathbb{C}} = SL(n;\mathbb{C})$ appear in **quiver varieties**:

$$0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \cdots \rightleftarrows \mathbb{C}^n.$$

Let M be the flat hyperkähler manifold

$$\bigoplus_{i=1}^{n-1} \mathbb{H}^{i(i+1)} = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1}) \oplus \mathrm{Hom}(\mathbb{C}^{i+1}, \mathbb{C}^i).$$

Then $M///U(1) \times U(2) \times \cdots \times U(n-1)$ can be identified with the **nilpotent cone** $\mathcal{N}$ in $\mathfrak{k}_{\mathbb{C}}$, which is the closure of the generic nilpotent coadjoint orbit in $\mathfrak{k}_{\mathbb{C}}$. If we shift the moment map by a suitable constant we get other coadjoint orbits in $\mathfrak{k}_{\mathbb{C}}$. So consider

$$Q = M///SU(1) \times SU(2) \times \cdots \times SU(n-1).$$

Q is stratified hyperkähler with dimension equal to $2(\dim K + \dim T)$ and a residual action of

$$(S^1)^{n-1} \times SU(n) \cong T \times K$$

which preserves the hyperkähler structure, and an action of $SU(2)$ which rotates the complex structures.

Properties of the universal hyperkähler implosion $Q = (T^* K_{\mathbb{C}})_{\text{hkimpl}}$:

1) Q is **stratified hyperkähler** with dimension equal to $2(\dim K + \dim T)$ and an action of

$$(S^1)^{n-1} \times SU(n) \cong T \times K$$

which preserves the hyperkähler structure, and an action of $SU(2)$ which rotates the complex structures.

2) The algebra of invariants $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$ is finitely generated and for any complex structure Q is the **complex symplectic quotient**

$$(K_{\mathbb{C}} \times \mathfrak{n}^0)//N$$

of $T^* K_{\mathbb{C}} = K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^*$ by N.

3) Q has a resolution of singularities

$$\tilde{Q} = K \times_T (\mathcal{T} \times \mathfrak{n})$$

where $\mathcal{T}$ is the hypertoric variety for T associated to the hyperplane arrangement given by the root planes in $\mathfrak{t}$.

4) $\quad Q = K\mathfrak{b}_+$ where
$\mathfrak{b}_+ = \{(\eta,\zeta) \in \mathcal{T} \times \mathfrak{n} : [\mu_j^T(\eta),\zeta] = 0, j = 1,2,3\}$
and $\mathcal{T}$ is the hypertoric variety as before.

5) Let $\mathcal{T}^\sigma = \{\eta \in \mathcal{T} : K_{\mu^T(\eta)} = K_\sigma\}$ for σ a face of $\mathfrak{t}_+$, and define $\mathfrak{b}_+^\sigma$ similarly. Then

$$Q = \bigsqcup_\sigma K \times_{K_\sigma} \mathfrak{b}_+^\sigma = \bigsqcup_\sigma (K \times_T (\mathcal{T}^\sigma \times \mathfrak{n}^{K_\sigma}))/[K_\sigma, K_\sigma].$$

6) Let $\{V_\varpi : \varpi \in \Pi\}$ be the set of fundamental representations of K. Then Q embeds in

$$H^0(\mathbb{P}^1, (\mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}) \otimes \mathcal{O}(2) \oplus \bigoplus_{\varpi,j} \wedge^j V_\varpi \otimes \mathcal{O}(j))$$

inducing a holomorphic and generically injective map from its twistor space to the line bundle $(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}) \otimes \mathcal{O}(2) \oplus \bigoplus_{\varpi,j} \wedge^j V_\varpi \otimes \mathcal{O}(j)$. The hyperkähler structure on Q can be recovered from this embedding.

7) The hyperkähler reduction at 0 of Q by T is the *nilpotent cone* $\mathcal{N}$ in $\mathfrak{k}_\mathbb{C}$; the reduction at a generic point of $\mathfrak{t} \otimes \mathbb{R}^3$ is a semisimple coadjoint orbit of $K_\mathbb{C}$.

8) Q can be described in terms of quivers, and also as a suitable moduli space of solutions to Nahm's equations.

9) If $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{t} \otimes \mathbb{R}^3 \subseteq \mathfrak{k} \otimes \mathbb{R}^3$ let

$$K_\zeta = K_{\zeta_1} \cap K_{\zeta_2} \cap K_{\zeta_3}.$$

Let $\mathcal{N}_\zeta$ be the nilpotent cone in $(K_\zeta)_{\mathbb{C}}$.
There is a $K \times$ SU(2)-equivariant embedding

$$\mathcal{N}_\zeta \hookrightarrow \mathfrak{k}_\zeta \otimes \mathbb{R}^3$$

whose composition with projection $\mathfrak{k}_\zeta \otimes \mathbb{R}^3 \to (\mathfrak{k}_\zeta)_{\mathbb{C}}$ is the inclusion of $\mathcal{N}_\zeta$ in $(\mathfrak{k}_\zeta)_{\mathbb{C}}$. If

$$\mathfrak{t}_{(+)} = \{\zeta + \xi \in \mathfrak{k} \otimes \mathbb{R}^3 | \zeta \in \mathfrak{t} \otimes \mathbb{R}^3 \text{ and } \xi \in \mathcal{N}_\zeta\}$$

and the hyperkähler implosion of X is

$$X_{\text{hkimpl}} = (X \times Q)///K,$$

then we have

$$X_{\text{hkimpl}} = \mu^{-1}(\mathfrak{t}_{(+)})/\sim$$

with $x \sim y \Leftrightarrow \mu(x) = \zeta + \xi$ and $\mu(y) = \zeta + \xi'$ where $\zeta \in \mathfrak{t} \otimes \mathbb{R}^3$ and $\xi, \xi' \in \mathcal{N}_\zeta$ and $x = ky$ for some $k \in [K_\zeta, K_\zeta]$.