Loop Groups and Characteristic Classes

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Outline

Characteristic Classes and Chern-Weil Theory

- $Q \to M$ a G-bundle
- A characteristic class is a cohomology class $c(Q) \in H^*(M)$ which is natural with respect to pull-backs:

$$f^* c(Q) = c(f^* Q)$$

- All characteristic classes are pulled-back from $H^*(BG)$:

$$c(Q) = f_Q^* c(EG)$$

- G compact, connected $\Rightarrow H^{2k}(BG) \simeq S^k(\mathfrak{g}^*)^G$

$$H^{2k}(BG) \ni f \longmapsto f(F, \ldots, F) \in H^{2k}(M)$$

Caloron Correspondence

There exists a bijection on isomorphism classes:

$$
\begin{tikzcd}
G \arrow[r] & \widetilde{P} \arrow[d] \\
& M \times S^1
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
LG \arrow[r] & P \arrow[d] \\
& M
\end{tikzcd}
$$

- $P_m = \Gamma\left(\widetilde{P}\big|_{\{m\}\times S^1}\right)$
- $\widetilde{P} = \dfrac{P \times G \times S^1}{LG}$

Caloron Correspondence

On the level of connections:

$$(\widetilde{P}, \widetilde{A}) \qquad \longleftrightarrow \qquad \big(P, (A, \Phi)\big)$$

- $\Phi \sim S^1$ part of $\widetilde{A}$

i.e. $$\widetilde{P}\big|_{S^1} \simeq G \times S^1 \implies \widetilde{A}\big|_{S^1} \sim \Theta + \Phi\, d\theta$$

- $\Phi \colon P \to L\mathfrak{g}$ satisfying

$$\Phi(p\gamma) = \mathrm{ad}(\gamma^{-1})\Phi(p) + \gamma^{-1}\partial\gamma$$

- $\widetilde{A} = \mathrm{ad}(g^{-1})A(\theta) + \Theta + \mathrm{ad}(g^{-1})\Phi\, d\theta$

String Classes

For $P \xrightarrow{LG} M$, define:

$$
\begin{array}{ccccc}
H^{2k}(BG) & \xrightarrow{\mathrm{cw}(\widetilde{P})} & H^{2k}(M \times S^1) & \xrightarrow{\int_{S^1}} & H^{2k-1}(M) \\
f & \longmapsto & f(\widetilde{F}^k) & \longmapsto & \displaystyle\int_{S^1} f(\widetilde{F}^k)
\end{array}
$$

- We have: $\widetilde{F} = \mathrm{ad}(g^{-1})\,(F + \nabla\Phi\, d\theta)$

  $\big(\text{where } \nabla\Phi = d\Phi + [A, \Phi] - \partial A\big)$

$$\implies \int_{S^1} f(\widetilde{F}^k) = k \int_{S^1} f(\nabla\Phi, F^{k-1})\, d\theta$$

String Classes

Proposition

$k \int_{S^1} f(\nabla\Phi, F^{k-1})\, d\theta$ is:

- *closed*
- *independent of choice of A and Φ*
- *natural*

We call

$$s_f(P) = k \int_{S^1} f(\nabla\Phi, F^{k-1})\, d\theta \in H^{2k-1}(M)$$

the string class of P associated to f

String Classes

Example (Murray–Stevenson (2003))

$k = 2, f = p_1 \in H^4(BG)$:

$$s_{p_1}(P) = \frac{-1}{4\pi^2} \int_{S^1} \langle \nabla \Phi, F \rangle \, d\theta \in H^3(M)$$

- Obstruction to lifting $P \xrightarrow{LG} M$ to $\widehat{P} \xrightarrow{\widehat{LG}} M$

Loop Groups and Classifying Spaces

Look at

1. ΩG

   (difficult caloron correspondence, easy classifying theory)
2. LG

   (easy caloron correspondence, difficult classifying theory)

String Classes for ΩG-bundles

Universal ΩG-bundle (Carey–Mickelsson (2000)):

- path fibration $PG \to G$

Classifying map:

- Solve $\Phi(p) = g^{-1}\partial g$
- $g = \mathrm{hol}_\Phi(p)$ – Higgs field holonomy

$$\begin{array}{ccc} P & \xrightarrow{\mathrm{hol}_\Phi} & PG \\ \downarrow & & \downarrow \\ M & \longrightarrow & G \end{array}$$

String Classes for ΩG-bundles

Choose a connection and Higgs field for $PG \to G \implies$

Proposition

$$s_f(PG) = \left(-\frac{1}{2}\right)^{k-1} \frac{k!(k-1)!}{(2k-1)!} f(\Theta, [\Theta, \Theta]^{k-1})$$

That is

$$s_f(PG) = \tau(f),$$

where $\tau \colon H^{2k}(BG) \to H^{2k-1}(G)$ is the transgression map

String Classes for ΩG-bundles

Theorem (Murray–V (2010))

If $P \to M$ is an ΩG-bundle and

$$s(P)\colon S^k(\mathfrak{g}^*)^G \to H^{2k-1}(M)$$

is the map which associates to any invariant polynomial f the string class of P (i.e. $s(P)(f) = s_f(P)$), then the following diagram commutes

$$\begin{tikzcd}
H^{2k}(BG) \arrow[r, "cw(\widetilde{P})"] \arrow[d, "\tau"'] \arrow[dr, "s(P)"] & H^{2k}(M \times S^1) \arrow[d, "\int_{S^1}"] \\
H^{2k-1}(G) \arrow[r, "\mathrm{hol}_\Phi^*"] & H^{2k-1}(M)
\end{tikzcd}$$

String Classes for LG-bundles

$P \to M$ an LG-bundle

- Want a model for $ELG \to BLG$ and a map $M \to BLG$

$LG \simeq \Omega G \rtimes G$ therefore take

$$\begin{array}{c} ELG = PG \times EG \\ \Big\downarrow \\ BLG = G \times_G EG \end{array}$$

So

$$H(BLG) = H(G \times_G EG) = H_G(G)$$

- Classifying map: $c_{LG} = (\mathrm{hol}_\Phi, c_G)$

String Classes for LG-bundles

So we expect

$$\begin{tikzcd}
H^{2k}(BG) \arrow[r, "cw(\widetilde{P})"] \arrow[d, dotted, "?"'] \arrow[dr, "s(P)"] & H^{2k}(M \times S^1) \arrow[d, "\int_{S^1}"] \\
H_G^{2k-1}(G) \arrow[r, "c_{LG}^*"] & H^{2k-1}(M)
\end{tikzcd}$$

Therefore we must calculate $s_f(ELG)$

Equivariant Cohomology

- G acts on X

Want to study $H(X/G)$

- Borel model:

$$H_G(X) = H(X \times_G EG)$$

- Cartan model:

$$\Omega_G(X) = \big(S(\mathfrak{g}^*) \otimes \Omega(X)\big)^G$$
$$(d_G\omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi))$$

$$H_G(X) = H(\Omega_G(X), d_G)$$

Borel model $\simeq$ Cartan model — Mathai–Quillen isomorphism

Equivariant Transgression Forms

Recall:
$$\tau(f) = \left(-\frac{1}{2}\right)^{k-1} \frac{k!(k-1)!}{(2k-1)!} f(\Theta, [\Theta, \Theta]^{k-1})$$
$$= -\int_0^1 f(F_t^k)$$

where $F_t = F(t\Theta) = \frac{1}{2}t^2[\Theta, \Theta]$

Define (Jeffrey (1995), Alekseev–Meinrenken (2009)):

$$\tau_G(f) = -\int_0^1 f\Big(\big(F_G(t\Theta)(\xi) + \xi\big)^k\Big)$$

where $F_G(t\Theta) = d_G(t\Theta) + \frac{1}{2}t^2[\Theta, \Theta]$

Universal String Class

Calculate $s_f(ELG)$:

- Choose A and Φ
- Calculate F and $\nabla\Phi$
- Calculate $k \int_{S^1} f(\nabla\Phi, F^{k-1})\, d\theta \in H^{2k-1}(G \times_G EG)$
- Apply Mathai–Quillen isomorphism

$$H^{2k-1}(G \times_G EG) \xrightarrow{\simeq} H^{2k-1}(\Omega_G(G))$$

Proposition

$$s_f(ELG) = \tau_G(f)$$

String Classes for LG-bundles

Theorem (V (arXiv:1005.4243))

If $P \to M$ is an LG-bundle and

$$s(P)\colon S^k(\mathfrak{g}^*)^G \to H^{2k-1}(M)$$

is the map which associates to any invariant polynomial f the string class of P (i.e. $s(P)(f) = s_f(P)$), then the following diagram commutes

$$\begin{tikzcd}
H^{2k}(BG) \arrow[rr, "cw(\widetilde{P})"] \arrow[d, "\tau_G"'] \arrow[drr, "s(P)"] & & H^{2k}(M \times S^1) \arrow[d, "\int_{S^1}"] \\
H^{2k-1}_G(G) \arrow[rr, "c^*_{LG}"] & & H^{2k-1}(M)
\end{tikzcd}$$