Bruhat order on involutions and combinatorics of coadjoint orbits

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Let G be a complex reductive algebraic group, T a maximal torus in G, B a Borel subgroup of G containing T, W the Weyl group of G with respect to T. For example, if $G = \mathrm{GL}_n(\mathbb{C})$, then one can put T to be the group of diagonal matrices and B to be the group of upper-triangular matrices. In this case, $W = S_n$, the symmetric group on n letters. Denote by $\mathcal{F} = G/B$ the flag variety. Let X_w be the Schubert subvariety of $\mathcal{F}$ corresponding to an element $w \in W$.

The Bruhat order on W plays a fundamental role in a multitude of contexts. For example, it encodes incidences among Schubert varieties, i.e., $X_v \subseteq X_w$ if and only if $v \leq w$. An interesting subposet of the Bruhat order is induced by the involutions, i.e., the elements of order 2 of W. We denote this subposet by $I(W)$. Activity around $I(W)$ was initiated by R. Richardson and T. Springer, who proved that the inverse Bruhat order on $I(S_{2n+1})$ encodes the incidences among the closed orbits of the action of the Borel subgroup of the special linear group on the symmetric variety $\mathrm{SL}_{2n+1}(\mathbb{C})/\mathrm{SO}_{2n+1}(\mathbb{C})$.

My goal is to involve coadjoint orbits into the picture. To each involution $w \in I(W)$ one can assign the coadjoint B-orbit Ω_w. It turns out that the Bruhat order encodes incidences among the closures of such orbits, i.e., $\Omega_v \subseteq \overline{\Omega}_w$ if and only if $v \leq w$. I will describe these connections between geometry of coadjoint orbits and combinatorial properties of $I(W)$. I will also formulate some open problems and conjectures.