HEISENBERG ODOMETERS

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Koopman unitary representation

Let $T = (T_g)_{g\in G}$ be an ergodic measure preserving action of a l.c.s.c. group G on a standard probability space $(X, \mathfrak{B}, \mu)$. Denote by $U_T = (U_T(g))_{g\in G}$ the associated Koopman unitary representation of G in $L^2(X, \mu)$:

$$U_T(g)f := f \circ T_g^{-1}, \qquad f \in L^2(X, \mu).$$

Actions with pure point spectrum. Abelian case

Suppose first that G is Abelian. If U_T is a direct countable sum of 1-dimensional unitary sub-representations then T is said *to have a pure point spectrum*. In 1932, J. von Neumann developed a theory of such actions in the case $G = \mathbb{R}$.

Three main aspects of this theory

- *isospectrality*: two ergodic flows with pure point spectrum are isomorphic if and only if the associated Koopman unitary representations are unitarily equivalent,
- *classification by simple algebraic invariants*: the ergodic flows with pure point spectrum considered up to isomorphism are in one-to-one correspondence with the countable subgroups in $\widehat{\mathbb{R}}$ which is the dual of $\mathbb{R}$,
- *structure*: if an ergodic flow has pure point spectrum then it is isomorphic to a flow by rotations on a compact metric Abelian group endowed with the Haar measure.

Remark

Similar results hold for the general Abelian G.

G. Mackey (1964) extended the concept of pure point spectrum to actions of non-Abelian groups: T has a pure point spectrum if U_T is a direct sum of countably many finite dimensional irreducible unitary representations of G.
He established a structure for these actions: an ergodic action T has pure point spectrum if and only if it is isomorphic to a G-action by rotations on a homogeneous space of a compact group. However, in general, the G-actions with pure point spectrum are not isospectral even in the case of finite G.

Objects considered in the talk

G is the 3-dimensional real Heisenberg group $H_3(\mathbb{R})$ which is apparently the 'simplest' non-Abelian nilpotent Lie group. Moreover, we single out a special class of actions of $H_3(\mathbb{R})$ which we call *odometers*. They are inverse limits of transitive $H_3(\mathbb{R})$-actions on homogeneous spaces by lattices in $H_3(\mathbb{R})$. For discrete finitely generated groups G, the G-odometers were considered by M. Cortez and S. Petit (2008) in the context of topological dynamics. We define G-odometers for arbitrary l.c.s.c. groups and study them as measure preserving dynamical systems. "Discrete" Heisenberg odometers, i.e. odometer actions of $H_3(\mathbb{Z})$ were considered earlier in ["The structure and the spectrum of Heisenberg odometers", S. Lightwood, A. Şahin and I. Ugarcovici, PAMS, to appear]

Our Purpose

To investigate whether von Neumann's theory of flows with pure point spectrum extends (or partially extends) to the Heisenberg odometers.

Heisenberg group $H_3(\mathbb{R})$

consists of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are arbitrary reals. The Heisenberg group endowed with the natural topology is a connected, simply-connected nilpotent Lie group.

Three homomorphisms

We now let

$$a(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, c(t) := \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\{a(t) \mid t \in \mathbb{R}\}$, $\{b(t) \mid t \in \mathbb{R}\}$ and $\{c(t) \mid t \in \mathbb{R}\}$ are three closed one-parameter subgroups in $H_3(\mathbb{R})$. The last is the center of $H_3(\mathbb{R})$. Every element g of $H_3(\mathbb{R})$ can be written uniquely as the product $g = c(t_3)b(t_2)a(t_1)$ for some $t_1, t_2, t_3 \in \mathbb{R}$.

Unitary dual of $H_3(\mathbb{R})$.

The set of unitarily equivalent classes of irreducible (weakly continuous) unitary representations of $H_3(\mathbb{R})$ is denoted by $\widehat{H_3(\mathbb{R})}$. The irreducible unitary representations of $H_3(\mathbb{R})$ are well known. They consist (up to unitary equivalence) of a family of 1-dimensional representations $\pi_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{R}$, and a family of infinite dimensional representations π_γ, $\gamma \in \mathbb{R} \setminus \{0\}$, as follows:

$$\pi_{\alpha,\beta}(c(t_3)b(t_2)a(t_1)) := e^{2\pi i(\alpha t_1 + \beta t_2)} \quad \text{and}$$
$$(\pi_\gamma(c(t_3)b(t_2)a(t_1))f)(x) := e^{2\pi i\gamma(t_3 + t_2 x)} f(x + t_1), \quad f \in L^2(\mathbb{R}, \lambda_{\mathbb{R}}).$$

Thus we can identify $\widehat{H_3(\mathbb{R})}$ with the disjoint union $\mathbb{R}^2 \sqcup \mathbb{R}^*$.

Lattices in $H_3(\mathbb{R})$. Invariants ξ_Γ, k_Γ and $p(\Gamma)$

Every lattice is co-compact.
Fix a lattice Γ in $H_3(\mathbb{R})$. There is a real $\xi_\Gamma > 0$ such that

$$\Gamma \cap \{c(t) \mid t \in \mathbb{R}\} = \{c(m\xi_\Gamma) \mid m \in \mathbb{Z}\}.$$

The commutator subgroup $[\Gamma, \Gamma]$ is of a finite index $k_\Gamma > 0$ in $\{c(t) \mid t \in \mathbb{R}\} \cap \Gamma$. The central extension

$$\{0\} \leftarrow \mathbb{R}^2 \xleftarrow{p} H_3(\mathbb{R}) \xleftarrow{c} \mathbb{R} \leftarrow \{0\}$$

induces a short exact sequence

$$\{0\} \longleftarrow p(\Gamma) \xleftarrow{p} \Gamma \xleftarrow{c} \xi_\Gamma \mathbb{Z} \longleftarrow \{0\}.$$

$p(\Gamma)$ is a lattice in $\mathbb{R}^2$.

Classification of lattices

Theorem

Given a lattice Γ in $H_3(\mathbb{R})$, there is an automorphism θ of $H_3(\mathbb{R})$ such that

$$\theta(\Gamma) = \left\{ \begin{pmatrix} 1 & l & \frac{n}{k_\Gamma} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, l, m, n \in \mathbb{Z} \right\}.$$

Hence two lattices Γ_1 and Γ_2 in $H_3(\mathbb{R})$ are automorphic if and only if $k_{\Gamma_1} = k_{\Gamma_2}$. Two lattices Γ_1 and Γ_2 in $H_3(\mathbb{R})$ are conjugate if and only if $k_{\Gamma_1} = k_{\Gamma_2}$ and $p(\Gamma_1) = p(\Gamma_2)$.

Odometer actions of locally compact groups

Let $\Gamma_1 \supset \Gamma_2 \supset \cdots$ be a nested sequence of lattices in G. Consider a projective sequence of homogeneous G-spaces

$$G/\Gamma_1 \leftarrow G/\Gamma_2 \leftarrow \cdots .$$

All arrows are G-equivariant and onto. Denote by X the projective limit of this sequence. Then X is a locally compact second countable G-space: G/Γ_1 is locally compact and every arrow is finite-to-one. X is compact if and only if each Γ_n is co-compact in G. The G-action is minimal and uniquely ergodic. The only invariant probability measure μ on X is the projective limit of the probability Haar measures on G/Γ_n.

Definition

We call the dynamical system (X, μ, G) a *G-odometer*.

Freeness of Heisenberg odometers

Theorem

Let T be the $H_3(\mathbb{R})$-odometer associated with a sequence $\Gamma_1 \supset \Gamma_2 \supset \cdots$ of lattices in $H_3(\mathbb{R})$. Then T is free if and only if $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$.

Example

Let $\Gamma_n := \{c(n!i_3)b(n!i_2)a(i_1) \mid i_1, i_2, i_3 \in \mathbb{Z}\}$. Then Γ_n is a lattice in $H_3(\mathbb{R})$, $\Gamma_1 \supset \Gamma_2 \supset \cdots$ and $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. On the other hand, $\bigcap_{n=1}^{\infty} \Gamma_n = \{a(i_1) \mid i_1 \in \mathbb{Z}\}$.

If Γ_n is normal in Γ_1 for each n and T is free then $\bigcap_{n=1}^{\infty} p(\Gamma_n) = \{0\}$.
In general, $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$ does not imply $\bigcap_{n=1}^{\infty} p(\Gamma_n) = \{0\}$.

Spectral analysis for transitive actions of $H_3(\mathbb{R})$ on nil-manifolds

Fix a lattice Γ in $H_3(\mathbb{R})$ and consider the homogeneous $H_3(\mathbb{R})$-space $H_3(\mathbb{R})/\Gamma$.

Let U denote the corresponding Koopman unitary representation of $H_3(\mathbb{R})$.

If $p(\Gamma) = A(\mathbb{Z}^2)$ for some matrix $A \in \mathrm{GL}_2(\mathbb{R})$ then we denote by $p(\Gamma)^*$ the *dual lattice* $(A^*)^{-1}\mathbb{Z}^2$ in $\mathbb{R}^2$. It is easy to see that the dual lattice does not depend on the choice of A.

Theorem

$$U = \bigoplus_{(\alpha,\beta)\in p(\Gamma)^*} \pi_{\alpha,\beta} \oplus \bigoplus_{0\neq n\in\mathbb{Z}} \bigoplus_{1}^{|n|k_\Gamma} \pi_{n\xi_\Gamma^{-1}}.$$

Corollary

Let Γ and Γ' be two lattices in $H_3(\mathbb{R})$. Denote by T and T' the corresponding measure preserving actions of $H_3(\mathbb{R})$ on the homogeneous spaces $H_3(\mathbb{R})/\Gamma$ and $H_3(\mathbb{R})/\Gamma'$ respectively. The following are equivalent:

- *T and T' are isomorphic.*
- *$p(\Gamma) = p(\Gamma')$ and $k_\Gamma = k_{\Gamma'}$.*
- *$p(\Gamma) = p(\Gamma')$ and $\xi_\Gamma = \xi_{\Gamma'}$.*
- *The Koopman representations of $H_3(\mathbb{R})$ generated by T and T' are unitarily equivalent.*
- *T and T' have the same maximal spectral type.*

Non-degenerate odometers

Denote by (X, μ, T) the $H_3(\mathbb{R})$-odometer associated with $\Gamma_1 \supset \Gamma_2 \supset \cdots$. Let (Y, ν) stand for the space of $(T_{c(t)})_{t\in\mathbb{R}}$-ergodic components and let $f : X \to Y$ stand for the corresponding projection. Then an $\mathbb{R}^2$-action $V = (V_{t_1,t_2})_{(t_1,t_2)\in\mathbb{R}^2}$ is well defined by the formula $V_{t_1,t_2} f(x) := f(T_{b(t_2)a(t_1)}x)$. We call it the *underlying* $\mathbb{R}^2$*-odometer.* It is the $\mathbb{R}^2$-odometer associated with the sequence $p(\Gamma_1) \supset p(\Gamma_2) \supset \cdots$ of lattices in $\mathbb{R}^2$.

Definition

We say that T is non-degenerate if one of the following equivalent conditions is satisfied:

- The underlying $\mathbb{R}^2$-odometer is non-transitive.
- The subgroup $\bigcup_{j=1}^{\infty} p(\Gamma_j)^*$ is not closed in $\mathbb{R}^2$
- The sequence $(p(\Gamma_j))_{j=1}^{\infty}$ does not stabilize, i.e. for each $j > 0$ there is $j_1 > j$ such that $p(\Gamma_j) \neq p(\Gamma_{j_1})$.

Spectral decomposition for Heisenberg odometers

Theorem

Let U stand for the Koopman unitary representation of $H_3(\mathbb{R})$ generated by a Heisenberg odometer T.

- *If T is non-degenerate then*

$$U = \bigoplus_{(\alpha,\beta)\in\bigcup_{j=1}^{\infty} p(\Gamma_j)^*} \pi_{\alpha,\beta} \oplus \bigoplus_{0\neq\gamma\in\bigcup_{j=1}^{\infty} \xi_{\Gamma_j}^{-1}\mathbb{Z}} \bigoplus_{1}^{\infty} \pi_\gamma.$$

- *If there is $l > 0$ such that $p(\Gamma_j) = p(\Gamma_l)$ for all $j \geq l$ then*

$$U = \bigoplus_{(\alpha,\beta)\in p(\Gamma_l)^*} \pi_{\alpha,\beta} \oplus \bigoplus_{0\neq\gamma\in\bigcup_{j=l}^{\infty} \xi_{\Gamma_j}^{-1}\mathbb{Z}} \bigoplus_{1}^{m(\gamma)} \pi_\gamma,$$

where $m(\gamma) := |\gamma|\xi_{\Gamma_j} k_{\Gamma_j}$ for each $\gamma \in \xi_{\Gamma_j}^{-1}\mathbb{Z}$, $j \geq l$.

Off-rational subgroups

Definition

A subgroup S in $\mathbb{R}^m$ is *off-rational* if its closure $\overline{S}$ is co-compact in $\mathbb{R}^m$ and there are a subgroup $Q \subset \mathbb{Q}^m$ and a matrix $A \in GL_m(\mathbb{R})$ such that $S = AQ$.

Given S, we associate to S an off-rational subgroup $\tau(S)$ in $\mathbb{R}$. Since S is off-rational, there is a sequence of matrices $A_j \in GL_m(\mathbb{R}) \cap M_m(\mathbb{Z})$ such that $A_1^{-1}\mathbb{Z}^m \subset A_2^{-1}\mathbb{Z}^m \subset \cdots$ and $\bigcup_{j=1}^{\infty} A_j^{-1}\mathbb{Z}^m = Q$ and hence $S = \bigcup_{j=1}^{\infty} AA_j^{-1}\mathbb{Z}^m$.Consider now a sequence of subgroups

$$\frac{\det A}{\det A_1}\mathbb{Z} \subset \frac{\det A}{\det A_2}\mathbb{Z} \subset \cdots$$

in $\mathbb{R}$. Then $\tau(S) := \bigcup_{j=1}^{\infty} \frac{\det A}{\det A_j}\mathbb{Z}$ is a dense off-rational subgroup of $\mathbb{R}$ if $m > 1$.

$\tau(S)$ does not depend on the choice of the sequence $(A_j)_{j=1}^{\infty}$.

Invariants S_Γ and ξ_Γ

Suppose we are given a sequence $\Gamma = (\Gamma_j)_{j=1}^{\infty}$ of lattices $\Gamma_1 \supset \Gamma_2 \supset \cdots$ in $H_3(\mathbb{R})$. Then $S_\Gamma := \bigcup_{j=1}^{\infty} p(\Gamma_j)^*$ is an off-rational subgroup of $\mathbb{R}^2$ and $\xi_\Gamma := \bigcup_{j=1}^{\infty} \xi_{\Gamma_j}^{-1}\mathbb{Z}$ is an off-rational subgroup in $\mathbb{R}$. If T is free then ξ_Γ is dense in $\mathbb{R}$.

Proposition

$\tau(S_\Gamma) \supset \xi_\Gamma$.

Theorem

Given an off-rational subgroup S in $\mathbb{R}^2$ and an off-rational subgroup ξ in $\mathbb{R}$ such that $\tau(S) \supset \xi$, there is a sequence Γ of lattices $\Gamma_1 \supset \Gamma_2 \supset \cdots$ in $H_3(\mathbb{R})$ such that $S_\Gamma = S$ and $\xi_\Gamma = \xi$. If S is dense then $\bigcap_{j=1}^{\infty} p(\Gamma_j) = \{0\}$. If, in addition, ξ is dense in $\mathbb{R}$ then $\bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$.

f-isomorphism

Definition

Two $H_3(\mathbb{R})$-odometers T and T' are called *f-isomorphic* if they are associated with some sequences $(\Gamma_j)_{j=1}^\infty$ and $(\Gamma'_j)_{j=1}^\infty$ (respectively) of lattices in $H_3(\mathbb{R})$ such that Γ_j and Γ'_j are conjugate in $H_3(\mathbb{R})$ for each j.

Theorem

- *Let $\Gamma = (\Gamma_j)_{j=1}^\infty$ and $\Gamma' = (\Gamma'_j)_{j=1}^\infty$ be two sequences of lattices in $H_3(\mathbb{R})$ such that $\Gamma_1 \supset \Gamma_2 \supset \cdots$ and $\Gamma'_1 \supset \Gamma'_2 \supset \cdots$. Let T denote the odometer associated to Γ and let T' denote the odometer associated to Γ'. Then T and T' are f-isomorphic if and only if $S_\Gamma = S_{\Gamma'}$ and $\xi_\Gamma = \xi_{\Gamma'}$.*
- *The Heisenberg odometers T and T' are f-isomorphic if and only if the Koopman unitary representations of $H_3(\mathbb{R})$ associated with them are unitarily equivalent.*

Heisenberg odometers are not isospectral

Example

(cf. [The structure and the spectrum of Heisenberg odometers, by S. Lightwood, A. Şahin and I. Ugarcovici, Example 4.9]). Fix a sequence of natural numbers $k_1 < k_2 < \cdots$ such that $k_1 = 1$ and $k_n(k_n + 1) = k_{n+1}$ for each n. Let

$$\Gamma_n := \{c(k_n j_3) b(k_n j_2) a(k_n j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\} \text{ and}$$
$$\Gamma_n' := \{c(k_n j_3 + j_1) b(k_n j_2) a(k_n j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\},$$

$n \in \mathbb{N}$. The corresponding $H_3(\mathbb{R})$-odometers T and T' are f-isomorphic but non-isomorphic. Let σ denote the flip in $H_3(\mathbb{R})$, i.e. $\sigma(a(t)) = b(t)$, $\sigma(b(t)) = a(t)$ and $\sigma(c(t)) = c(-t)$, $t \in \mathbb{R}$. Moreover, T is symmetric, i.e. T isomorphic to $T \circ \sigma$ but T' is asymmetric. Nevertheless, T' is f-isomorphic to $T' \circ \sigma$.

Product of odometers

Theorem

Let T and T' be two Heisenberg odometers associated with the nested sequences of lattices $\Gamma = (\Gamma_j)_{j=1}^\infty$ and $\Gamma' = (\Gamma'_j)_{j=1}^\infty$ in $H_3(\mathbb{R})$ respectively. Then

- $T \times T'$ *is ergodic if and only if* $S_\Gamma \cap S_{\Gamma'} = \{0\}$.
- $T \times T'$ *is ergodic and has discrete maximal spectral type if and only if* $S_\Gamma \cap S_{\Gamma'} = \{0\}$ *and* $\xi_\Gamma \cap \xi_{\Gamma'} = \{0\}$. *In this case the Koopman unitary representation* $U_{T\times T'}$ *of* $H_3(\mathbb{R})$ *decomposes into irreducible representations as follows*

$$U_{T\times T'} = \bigoplus_{(\alpha,\beta)\in S_\Gamma + S_{\Gamma'}} \pi_{\alpha,\beta} \oplus \bigoplus_{0\neq\gamma\in\xi_\Gamma+\xi_{\Gamma'}} \bigoplus_1^\infty \pi_\gamma$$

- $T \times T'$ *is not isomorphic (even not spectrally equivalent) to any Heisenberg odometer.*

Let $\Gamma = \{c(n/k)b(m)a(l) \mid n, m, l \in \mathbb{Z}\}$ for some $k \in \mathbb{N}$. Every element $g \in H_3(\mathbb{R})$ can be written uniquely as $g = c(t_3)b(t_2)a(t_1)\gamma$ for some $\gamma \in \Gamma$ and $0 \le t_3 < 1/k$, $0 \le t_2 < 1$ and $0 \le t_1 < 1$. Hence the quotient space $H_3(\mathbb{R})/\Gamma$ is a 3-torus

$$\mathbb{T}^3 = \{(t_1, t_2, t_3) \mid 0 \le t_1 < 1, 0 \le t_2 < 1 \text{ and } 0 \le t_3 < 1/k\}.$$

We write the $H_3(\mathbb{R})$-action on the homogeneous space $H_3(\mathbb{R})/\Gamma$ in a skew product form:

$$T_g(y, z) = (p(g) \cdot y, \alpha(g, y) + z),$$

where $(y, z) \in Y \times Z := (\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{R}/k^{-1}\mathbb{Z})$, the symbol "$\cdot$" denotes the usual action of $\mathbb{R}^2$ on Y by rotations and $\alpha : H_3(\mathbb{R}) \times Y \to Z$ is the corresponding cocycle.

Let Δ_d denote the measure on $Y \times Y$ sitting on the subset $\{(y, d+y) \mid y \in Y\}$ and projecting on the Haar measure on Y along each of the two coordinate projections. Given a closed subgroup Λ in $Z \times Z$, we denote by λ_Λ the Haar measure on Λ. We consider it as a measure on $Z \times Z$. Given $z \in Z$, we denote by $\lambda_\Lambda \circ z$ the image of λ_Λ under the rotation $Z \times Z \ni (z_1, z_2) \mapsto (z_1, z_2 + z) \in Z \times Z$. Let $D_q := \bigcup_{j=0}^{q-1} \{(t + k^{-1}\mathbb{Z}, t + j/(qk) + k^{-1}\mathbb{Z} \mid 0 \le t < 1\} \subset Z \times Z$.

Theorem

The set $J_2^e(T)$ of all ergodic 2-fold self-joinings of T is the union of two families as follows:

$$\{\Delta_d \times \lambda_{Z\times Z} \mid d \text{ is aperiodic}\} \cup \{\Delta_d \times \lambda_{D_{q(d)}} \circ z \mid d \text{ is periodic, } z \in Z\}.$$

Every joining from the first family is a non-transitive dynamical system and every joining from the second family is a transitive dynamical system.

Remark

There exist ergodic 2-fold self-joinings of T which are not isomorphic to any Heisenberg odometer.

(B) Self-joinings of general Heisenberg odometers

We now consider a Heisenberg odometer T associated to a sequence of latices $\Gamma_1 \supset \Gamma_2 \supset \cdots$. The T-action can be represented as a skew product. The space of this action is the product $Y \times Z$ of two compact Abelian groups $Y := \operatorname{proj\,lim}_{j\to\infty} \mathbb{R}^2/p(\Gamma_j)$ and $Z := \operatorname{proj\,lim}_{j\to\infty} Z_j$, where $Z_j := \mathbb{R}/\xi_{\Gamma_j}\mathbb{Z}$. Given $d \in Y$, we denote by Δ_d the image of the Haar measure on Y under the map $Y \ni y \mapsto (y, y+d) \in Y \times Y$. Every element $d \in Y$ is a sequence $(d_j)_{j\in\mathbb{N}}$ of elements $d_j \in \mathbb{R}^2/p(\Gamma_j)$ such that d_{j+1} maps to d_j under the natural projection $\mathbb{R}^2/p(\Gamma_{j+1}) \to \mathbb{R}^2/p(\Gamma_j)$ for each j. In a similar way, every element $z \in Z$ is a sequence $(z_j)_{j\in\mathbb{N}}$ of elements $z_j \in Z_j$ such that z_{j+1} maps to z_j under the natural projection $Z_{j+1} \to Z_j$ for each j.

If d_j is periodic then we denote by D_j be the closed subgroup of $Z_j \times Z_j$ associated with d_j in the way described in the part (A). We note that D_j contains the diagonal of $Z_j \times Z_j$ as a subgroup of finite index. Moreover, D_{j+1} maps onto D_j under the natural projection $Z_{j+1} \to Z_j$ for each j. Hence a projective limit $D_d := \operatorname{proj}\lim_{j\to\infty} D_j$ is well defined. It is a closed subgroup of Z. Given a closed subgroup Λ of $Z \times Z$, let λ_Λ stand for the Haar measure on Λ. Given $z \in Z$, let $\lambda_\Lambda \circ z$ denote the image of λ_Λ viewed as a measure on $Z \times Z$ under the rotation $(z_1, z_2) \mapsto (z_1, z_2 + z)$ of $Z \times Z$.

Theorem

The set $J_2^e(T)$ of all ergodic 2-fold self-joinings of T is the union of the following two families:

$$J_2^e(T) = \{\Delta_d \times \lambda_{Z\times Z} \mid \ d = (d_j)_{j\in\mathbb{N}} \text{ with } d_j \text{ aperiodic for each } j\}$$
$$\cup \{\Delta_d \times \lambda_{D_d} \circ z \mid d = (d_j)_{j\in\mathbb{N}} \text{ with } d_j \text{ periodic for each } j \text{ and } z \in Z\}.$$

Let T be an ergodic action of $H_3(\mathbb{R})$ on a standard probability space (X, μ). Denote by U_T the corresponding Koopman unitary representation of $H_3(\mathbb{R})$.

Theorem

If U_T is unitarily equivalent to the Koopman unitary representation generated by the action Q of $H_3(\mathbb{R})$ by translations on $H_3(\mathbb{R})/\Gamma$ for a lattice Γ in $H_3(\mathbb{R})$ then T is isomorphic to Q.

Thus the class of transitive Heisenberg odometers is spectrally determined.

Theorem

- *The subclass of degenerate Heisenberg odometers is spectrally determined.*
- *Let T be a non-degenerate Heisenberg odometer. Then there is an ergodic action R of $H_3(\mathbb{R})$ such that R has the same maximal spectral type as T but R is not isomorphic to T (and hence to any $H_3(\mathbb{R})$-odometer).*
- *There is is an ergodic action of $H_3(\mathbb{R})$ which is unitarily equivalent to a Heisenberg odometer but which is not isomorphic to any Heisenberg odometer.*

On $H_3(\mathbb{Z})$-odometers (considered in "The structure and the spectrum of Heisenberg odometers", by S. Lightwood, A. Șahin and I. Ugarcovici, PAMS, to appear)

Let $\Gamma_1 \supset \Gamma_2 \supset \cdots$ be a decreasing sequence of cofinite subgroups in $H_3(\mathbb{Z})$. Denote by $T = (T_g)_{g \in H_3(\mathbb{Z})}$ the associated $H_3(\mathbb{Z})$-odometer. Let (X, μ) be the space of this odometer.
We call T *normal* if Γ_j is normal in $H_3(\mathbb{Z})$ for each j.
Let $T' = (T'_g)_{g \in H_3(\mathbb{R})}$ denote the $H_3(\mathbb{R})$-odometer associated with $\Gamma_1 \supset \Gamma_2 \supset \cdots$. Then T' is the action induced from T.

If T is normal then X is a compact totally disconnected group and μ is the normalized Haar measure on X. Indeed, we obtain a sequence

$$H_3(\mathbb{Z})/\Gamma_1 \leftarrow H_3(\mathbb{Z})/\Gamma_2 \leftarrow \cdots$$

of finite groups $H_3(\mathbb{Z})/\Gamma_j$ and canonical onto homomorphisms such that $X = \operatorname{proj}\lim_{j\to\infty} H_3(\mathbb{Z})/\Gamma_j$. Moreover, a group homomorphism $\varphi : H_3(\mathbb{Z}) \to X$ is well defined by the formula $\varphi(g) = (\varphi(g)_j)_{j=1}^{\infty}$, where $\varphi(g)_j := g\Gamma_j$. Of course, $\varphi(H_3(\mathbb{Z}))$ is dense in X. It is easy to see that $T_g x = \varphi(g)x$ for all $g \in H_3(\mathbb{Z})$ and $x \in X$. Hence T has a pure point spectrum in the sense of G. Mackey. Moreover, T is normal in the sense of R.Zimmer (1976).

Theorem

The normal $H_3(\mathbb{Z})$-odometers are isospectral.

Let L_j denote the left regular representation of $H_3(\mathbb{Z})/\Gamma_j$. Let $\mathcal{I}_j$ stand for the unitary dual of $H_3(\mathbb{Z})/\Gamma_j$. It is well known that (up to the unitary equivalence) $L_j = \bigoplus_{\tau\in\mathcal{I}_j}\bigoplus_1^{d_\tau}\tau$, where d_τ is the dimension of τ. In particular, $\#(H_3(\mathbb{Z})/\Gamma_j) = \sum_{\tau\in\mathcal{I}_j} d_\tau^2$. Moreover, $\#\mathcal{I}_j$ equals the cardinality of the set of congugacy classes in $H_3(\mathbb{Z})/\Gamma_j$.

Spectral decomposition of the normal $H_3(\mathbb{Z})$-odomoters

The canonical projection $X \to H_3(\mathbb{Z}/\Gamma_j)$ generates an embedding $L^2(H_3(\mathbb{Z})/\Gamma_j) \subset L^2(X)$. Therefore we obtain an increasing sequence

$$L^2(H_3(\mathbb{Z})/\Gamma_1) \subset L^2(H_3(\mathbb{Z})/\Gamma_2) \subset \cdots$$

of U_T-invariant subspaces whose union is dense in $L^2(X)$ and such that the restriction $U_T \restriction L^2(H_3(\mathbb{Z})/\Gamma_j)$ is unitarily equivalent to $L_j \circ p_j$, where $p_j : H_3(\mathbb{Z}) \to H_3(\mathbb{Z})/\Gamma_j$ is the canonical projection.

Theorem

Let $\mathcal{I}_T := \bigcup_{j\in\mathbb{N}}\{\tau \circ p_j \mid \tau \in \mathcal{I}_j\}$ and d_ι is the dimension of ι. Then we have

$$U_T = \bigoplus_{\iota\in\mathcal{I}_T} \bigoplus_{1}^{d_\iota} \iota.$$

An explicit computation of $\mathcal{I}_T$ in terms of the sequence $(\Gamma_j)_{j=1}^{\infty}$ was done in [Lightwood, A. Şahin and I. Ugarcovici, to appear].

Corollary

Two normal $H_3(\mathbb{Z})$-odometers T and R are (measure theoretically) isomorphic if and only if $\mathcal{I}_T = \mathcal{I}_R$.

Remark

There are non-isomorphic normal $H_3(\mathbb{Z})$-odometers such that the Koopman representations of $H_3(\mathbb{R})$ generated by the $H_3(\mathbb{R})$-odometers associated with the same sequences of lattices are unitarily equivalent.

Questions

Let Γ be a lattice in a l.c.s.c. group G.

Definition

Two ergodic actions T and R of Γ are called flow equivalent if the induced actions $\mathrm{Ind}_\Gamma^G(T)$ and $\mathrm{Ind}_\Gamma^G(T)$ are isomorphic.

If G is Abelian then the actions are flow equivalent if and only if they are isomorphic.

Questions

- Are there flow equivalent non-isomorphic ergodic actions of $H_3(\mathbb{Z})$?
- The same within the class of odometers?