Invariant measures on the circle

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For an integer $N \geq 1$ consider the endomorphism φ_N of the unit circle $\mathbb{T}$ given by $\varphi_N(\zeta) = \zeta^N$. It is known that besides Haar measure there are many φ_N-invariant atomless probability measures on $\mathbb{T}$, see [B].

There are natural ways to characterize a measure μ on $\mathbb{T}$ by an associated function defined either on $\mathbb{T}$ or holomorphic in the unit disc. Invariance of the measure under φ_N translates into functional equations for the corresponding functions. For example consider the holomorphic function $f_\mu = \exp(-h_\mu)$ on the unit disc D where h_μ is the Herglotz-transform of μ

$$h_\mu(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \, .$$

Then φ_N-invariance of μ is equivalent to a functional equation for $f = f_\mu$

$$f(z^N)^N = \prod_{\zeta^N = 1} f(\zeta z) \, . \tag{1}$$

Theorem 1. Up to a unique positive constant any non-zero function f in $\mathcal{N}$ satisfying (1) is a quotient of singular inner functions.

Thus Blaschke products and outer functions in $\mathcal{N}$ cannot satisfy (1) unless they are constant.

Not much is known about measures on $\mathbb{T}$ which are invariant under at least two endomorphisms φ_N and φ_M with N prime to M, but see [R]. It is therefore interesting to look for holomorphic functions on D which satisfy the functional equation (1) for several integers N. Consider the multiplicative monoid $\mathcal{S}$ generated by pairwise prime integers $N_1, \ldots, N_s \geq 2$. It acts on $\mathbb{T}$ if we identify $N \in \mathcal{S}$ with φ_N. For a subgroup $\mathcal{G} \subset \mathcal{O}^1 = \{f \in \mathcal{O}(D)^\times \mid f(0) = 1\}$ set

$$H^0(\mathcal{S}, \mathcal{G}) = \{f \in \mathcal{G} \mid f \text{ satisfies (1) for all } N \in \mathcal{S}\}$$

and

$$Z(\mathcal{S}, \mathcal{G}) = \{\alpha \in \mathcal{G} \mid \prod_{\zeta^N = 1} \alpha(\zeta z) = 1 \quad \text{for } 1 \neq N \in \mathcal{S}\} .$$

Here the conditions need to be checked for $N = N_1, \ldots, N_s$ only. The group $Z(\mathcal{S}, \mathcal{O}^1)$ is easy to describe as a certain quotient of $\mathcal{O}^1$. Moreover there are mutually inverse isomorphisms

$$Z(\mathcal{S}, \mathcal{O}^1) \underset{\Phi_{\mathcal{S}}}{\overset{\Psi_{\mathcal{S}}}{\rightleftarrows}} H^0(\mathcal{S}, \mathcal{O}^1) .$$

For $s = 1$ they are given by the formulas

$$\Phi_{\mathcal{S}}(f)(z) = f(z)/f(z^{N_1}) \quad \text{and} \quad \Psi_{\mathcal{S}}(\alpha)(z) = \prod_{\nu=0}^{\infty} \alpha(z^{N_1^\nu}) .$$

For general $\mathcal{S}$ we have

$$\Psi_{\mathcal{S}}(\alpha) = \prod_{N \in \mathcal{S}} \alpha(z^N) .$$

Thus for $f \in \mathcal{O}^1$ the description of simultanous solutions of (1) is easy. The situation becomes interesting when one imposes growth conditions on the solutions f. Recall that for a probability measure μ on $\mathbb{T}$ the function f_μ lies in the Hardy space $H^\infty(D)$ of bounded analytic functions on D.

If μ is φ_N-invariant for $N \in \mathcal{S}$ then f_μ lies in $H^0(\mathcal{S}, \mathcal{O}^1)$. Consequently $\Phi_{\mathcal{S}}(f_\mu) \in Z(\mathcal{S}, \mathcal{N}^1)$ where $\mathcal{N}^1 = \mathcal{N}^\times \cap \mathcal{U}$. Here we have used that quotients of nowhere vanishing bounded holomorphic functions lie in $\mathcal{N}^\times$.

It is not known which functions are of the form f_μ for an $\mathcal{S}$-invariant probability measure μ. By the above they can be recovered from $\Phi_{\mathcal{S}}(f_\mu)$ by applying $\Psi_{\mathcal{S}}$. Thus it is natural to study the map $\Psi_{\mathcal{S}}$ on $Z(\mathcal{S}, \mathcal{N}^1)$. The space $Z(\mathcal{S}, \mathcal{N}^1)$ is naturally a quotient of $\mathcal{N}^1$ with a known kernel. The image under $\Psi_{\mathcal{S}}$ contains the space $H^0(\mathcal{S}, \mathcal{N}^1)$ whose structure we would like to understand but it is strictly bigger. One basic result is the following

Theorem 2. There is an inclusion $\quad \Psi_{\mathcal{S}}(Z(\mathcal{S},\mathcal{N}^1)) \subset H^0(\mathcal{S},\mathcal{N}^1_s)$.

Here $\mathcal{N}^1_s = \mathcal{N}^\times_s \cap \mathcal{U}$ and $\mathcal{N}_s$ is the algebra of functions $f \subset \mathcal{O}(D)$ that can be written in the form $f = g_1 g_2^{-1}$ where g_2 has no zeroes and both g_1 and g_2 satisfy an estimate of the form

$$|g(z)| \leq a_g \exp(r_g \log^s(1-|z|)^{-1}) \quad \text{for } z \in D \tag{2}$$

where $a_g \geq 0$ and $r_g \geq 0$ are constants. For $s = 0$ the estimate (2) asserts that $g \in H^\infty(D)$ so that $\mathcal{N}_0 = \mathcal{N}$. For $s = 1$ it asserts that

$$|g(z)| \leq a_g(1-|z|)^{-r_g} \ .$$

This means that $g \in \mathcal{A}^{-\infty}$ in the notation of Korenblum [K1], [K2]. The more general classes $\mathcal{N}_s$ appear in the works [BL], [K4] and [S] for example.

Classically the elements of $\mathcal{N}^1$ can be described by finite signed measures on $\mathbb{T}$. More generally, by a theorem of Korenblum the elements of $\mathcal{N}^1_s$ correspond to real premeasures of bounded κ_s-variation on the circle. Here κ_s is the generalized entropy-function on $[0,1]$

$$\kappa_s(x) = x \sum_{\nu=0}^{s} \frac{1}{\nu!} |\log x|^\nu \ .$$

Thus $\kappa_0(x) = x$ and $\kappa_1(x) = x(1+|\log x|) = x \log \frac{e}{x}$. The premeasure μ on $\mathbb{T}$ is of bounded κ_s-variation if there is a constant $A \geq 0$ such that

$$\sum_j |\mu(C_j)| \leq A \sum_j \kappa_s(|C_j|)$$

holds for all finite partitions of $\mathbb{T}$ into disjoint connected subsets C_j (arcs). Here $|C|$ is the arc length of C normalized by $|\mathbb{T}| = 1$.

If the premeasure μ corresponds to $f \in \mathcal{N}^1_s$ then as for measures, μ is φ_N-invariant if and only if f satisfies equation (1). Hence we have obtained an injection from $Z(\mathcal{S},\mathcal{N}^1)$ into the space of premeasures of bounded κ_s-variation which are invariant under $N_1,\ldots,N_s$. One can do a little better: For suitable functions in $Z(\mathcal{S},\mathcal{N}^1)$ one even obtains premeasures of bounded κ_{s-1}-variation invariant under $N_1,\ldots,N_s$.

Classically the atoms of a measure μ can be seen in the function f_μ. For the Korenblum correspondence between premeasures and functions this is still

true but more subtle. It rests on a positivity argument as with the Féjèr kernel in Fourier analysis.
In the theory described up to now there are analogous assertions for spaces of atomless (pre-)measures and functions. For example, one obtains many φ_N and φ_M invariant atomless premeasures of bounded κ_1-variation.

As part of a more general theory, Korenblum has shown that premeasures μ of $\kappa = \kappa_s$-bounded variation induce compatible measures μ^F on the Borel algebras of κ-Carleson sets F. These are closed subsets of $\mathbb{T}$ of Lebesgue measure zero such that

$$\sum_I \kappa(|I|) < \infty \ .$$

Here $\mathbb{T} \setminus F = \amalg I$ is the decomposition into connected components I. The family $\mu_s = (\mu^F)$ is called the κ-singular measure attached to μ. Using Korenblums results and general facts from measure theory we show that κ-singular measures can be interpreted as "κ-thin measures" $\tilde{\mu}$. These live in the Grothendieck group of a semigroup of positive σ-finite measures on the Borel algebra of $\mathbb{T}$ (with further properties). Thus $\tilde{\mu}$ is given by a class of pairs of σ-finite positive measures $\tilde{\mu}_i$:

$$\tilde{\mu} = [\tilde{\mu}_1, \tilde{\mu}_2] \ .$$

Because of a cancellation property there is equality

$$[\tilde{\mu}_1, \tilde{\mu}_2] = [\tilde{\nu}_1, \tilde{\nu}_2]$$

if and only if $\tilde{\mu}_1 + \tilde{\nu}_2 = \tilde{\mu}_2 + \tilde{\nu}_1$. Combining the previously defined maps $\Psi_{\mathcal{S}}$ with the passage to κ_s-thin measures, we obtain for every $\alpha \in Z(\mathcal{S}, \mathcal{N}^1)$ or corresponding measure σ, pairs of σ-finite measures $\tilde{\mu}_1, \tilde{\mu}_2$ with $\tilde{\mu}_1 + N_* \tilde{\mu}_2 = N_* \tilde{\mu}_1 + \tilde{\mu}_2$ for all $N \in \mathcal{S}$. The measures $\tilde{\mu}_i \geq 0$ live on countable unions of κ_s-Carleson sets and are restricted by further properties. If $\tilde{\mu}_1$ or $\tilde{\mu}_2$ is finite then $\tilde{\mu} = \tilde{\mu}_1 - \tilde{\mu}_2$ is a signed measure and both $\tilde{\mu}^+$ and $\tilde{\mu}^-$ are $\mathcal{S}$-invariant.

We prove that every $\mathcal{S}$-invariant positive ergodic probability measure which is non-zero on some κ_s-Carleson set is κ_s-thin and can be obtained by the preceeding constructions. The last condition may be automatically satisfied. This is true if non-constant cyclic elements in certain topological algebras $\mathcal{A}_\gamma \subset \mathcal{O}(D)$ defined by growth conditions cannot satisfy the functional equation (1) for too many coprime integers N. The relation comes from

Korenblum's theory [K2], [K3] characterizing cyclicity in terms of vanishing κ-singular measure. For $\gamma = 0$ the assertion is true.

There are some useful operations on functions, (pre-)measures and (Schwartz-)distributions. These operations behave like Frobenius, Verschiebung and the Teichmüller character for Witt vectors. In fact the ring $\mathcal{D}'(\mathbb{T})$ of distributions on $\mathbb{T}$ under convolution embeds naturally into the ring of big Witt vectors of $\mathbb{C}$ such that the corresponding operations on both sides are identified. As a small example we note that the Artin–Hasse exponential for the prime p is the image of a p-invariant premeasure on $\mathbb{T}$ of κ_1-bounded variation which is not a measure and whose κ_1-thin (or singular) measure is zero.

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