SIMPLE-MINDED ENTROPY FOR UP TO AMENABLE ACTIONS

This note shows how I was gradually discovering the validity of the equality between the entropy of a process $h(G,\mathcal{P})$ and a simplified notion, which I denoted $h^*(G,\mathcal{P})$, for actions of more and more complex countable groups. I started with $\mathbb{Z}$, then $\mathbb{Z}^2$, then I passed to amenable right-orderable groups, exercised cyclic groups, to finally reach general amenable groups. Of course, the last proof overrides everything that precedes it, but I keep everything as a record of my way through.

The "simple minded formula" fails for free groups (e.g. for F_2) in the sense that it produces a parameter which is not an isomorphism invariant. In this sense it yields to sofic entropy.

So, I proposed two variants (denoted $h^{**}(G,\mathcal{P})$ and $h^{***}(G,\mathcal{P})$), which are always going to be isomorphism invariants, however, we I not know whether they behave well for Bernoulli shifts, i.e., whether for such processes they are equal to the static entropy of the independent generator.

This subject was presented at Max Planck Institute in Bonn during the activity "Dynamics and Numbers" (June 17, 2014).

After the presentation, I talked to Benjy Weiss, and (how disappointingly for me) it turned out that he knew all about it since quite long. He even knew that the variant $h^{***}(G,\mathcal{P})$ works (behaves well for Benoulli shifts) for actions of sofic groups. (Since this result is currently under preparation, I cannot give any reference.)

So, here we go. Below I present perhaps independent proofs of known facts. But nothing really new.

It seems, that the field remains open for topological entropy. At least, Benjy Weiss believes so. The static entropy of an open cover $\mathbf{H}(\mathcal{U}^n)$ (defined as log of the cardinality of the smallest subcover) is not strongly subadditive (the numbers $r(n,\epsilon)$, or $s(n,\epsilon)$, i.e., the log's of the cardinalities of maximal (n,ϵ)-seprated or minimal (n,ϵ)-spanning sets, used to define topological entropy, are not even subadditive), so the proofs do not pass directly. Probably, for one open cover the analogous notion (say $\mathbf{h}^*(G,\mathcal{U})$) does not equal the traditional notion $\mathbf{h}(G,\mathcal{U})$ even for $\mathbb{Z}$-actions, but there is hope that after taking the supremum over all open covers we will get equality. It is also interesting to see whether topological entropy can be defined with help of some strongly subadditive notion replacing the imperfect $\mathbf{H}(\mathcal{U}^n)$ or $r(n,\epsilon)$, or $s(n,\epsilon)$. This is subject of some current efforts of my research group.

1. $\mathbb{Z}$-actions

We consider a measure preserving transformation (X,Σ,μ,T), with a finite (or countable) measurable partition $\mathcal{P}$ of X. If $F\subset\mathbb{Z}$ then by $\mathcal{P}^F$ we denote $\bigvee_{n\in F}T^{-n}(\mathcal{P})$.

Definition 1.1.

$$h^*(T,\mathcal{P})=\inf\{\tfrac{1}{|F|}H(\mathcal{P}^F):\ |F|<\infty\}.$$

Theorem 1.2.

$$h^*(T,\mathcal{P}) = h(T,P).$$

Proof. We have

$$h(T,\mathcal{P}) = \inf_n \tfrac{1}{n} H(P^{[0,n-1]}) = \inf\{\tfrac{1}{|F|} H(\mathcal{P}^F) :\ F = [0,n-1], n \in \mathbb{N}\} \geq h^*(T,\mathcal{P}).$$

For the converse, for any finite set $F = \{n_1, n_2, \dots, n_k\}$, we write

$$H(\mathcal{P}^F) = H(\mathcal{P}^{\{n_1\}}) + H(\mathcal{P}^{\{n_2\}}|\mathcal{P}^{\{n_1\}}) + H(\mathcal{P}^{\{n_3\}}|\mathcal{P}^{\{n_1,n_2\}}) + \cdots$$
$$\cdots + H(\mathcal{P}^{\{n_k\}}|\mathcal{P}^{\{n_1,n_2,\dots,n_{k-1}\}}),$$

hence $\frac{1}{|F|} H(\mathcal{P}^F)$, being an average of the terms on the right, is not smaller than the smallest term. I.e., there exists $i \in \{1, 2, \dots, k\}$ such that

$$\tfrac{1}{|F|} H(\mathcal{P}^F) \geq H(\mathcal{P}^{\{n_i\}}|\mathcal{P}^{\{n_1,n_2,\dots,n_{i-1}\}}).$$

By invariance of the measure, the right hand side equals

$$H(\mathcal{P}|\mathcal{P}^{\{n_1-n_i,n_2-n_i,\dots,n_{i-1}-n_i\}}).$$

Since all exponents on the right are strictly negative, this expression is not smaller than $H(\mathcal{P}|\mathcal{P}^{\{-1,-2,\dots\}}) = h(T,\mathcal{P})$. Application of the infimum over all finite sets F completes the proof. □

Remark 1.3. The entropy $h^*(T,\mathcal{P})$ can be written as

$$h^*(T,\mathcal{P}) = \inf_n \tfrac{1}{n} \inf\{H(\mathcal{P}^F) :\ |F| = n\} =: \inf_n \tfrac{1}{n} H^*(n,\mathcal{P}).$$

Now, one can prove that the sequence $H^*(n,\mathcal{P})$ is subadditive (see below), hence the infimumm over n can be written as a limit or upper limit, according to preference.

Proof of subadditivity.

$$H^*(n+m,\mathcal{P}) = \inf\{H(\mathcal{P}^F) :\ |F| = n+m\} =$$
$$\inf\{H(\mathcal{P}^{F\cup E}) :\ |F| = n, |E| = m, F \cap E = \emptyset\} \leq$$
$$\inf\{H(\mathcal{P}^F) + H(\mathcal{P}^E) :\ |F| = n, |E| = m, F \cap E = \emptyset\} =$$
$$\inf\{H(\mathcal{P}^F) + H(\mathcal{P}^E) :\ |F| = n, |E| = m\},$$

where the last equality follows by invariance and the fact that any finite sets can be shifted to disjoint positions. Now, since the infimum involves two independent variables F and E and two expressions each depending on only one of them, the infimum equals the sum of infima, which, by definition, equals $H^*(n,\mathcal{P}) + H^*(m,\mathcal{P})$. □

By the way, it is completely clear that for Bernoulli shifts (on any countable group) we have $h^*(G,\mathcal{P}) = H(\mathcal{P})$.

2. $\mathbb{Z}^2$-actions

We will now prove the analog of Theroem 1.2 for $\mathbb{Z}^2$-actions. We will use the following notation: The action has two generating maps, S and T. Finite sets F are now subsets of $\mathbb{Z}^2$.

Theorem 2.1. *Define, as before,*

$$h^*(S,T,\mathcal{P}) = \inf\{\tfrac{1}{|F|}H(\mathcal{P}^F) : \ |F| < \infty\}.$$

Then

$$h^*(S,T,\mathcal{P}) = h(S,T,\mathcal{P}).$$

By $[0, m-1] \times [0, n-1]$ we will denote the rectangle in $\mathbb{Z}^2$, starting at $(0,0)$, with dimensions $m \times n$. We will imagine that m is the number of columns and that it describes the number of iterates of the transformation S. The proof is based on the following observation:

Lemma 2.2.

$$h(S,T,\mathcal{P}) = H(\mathcal{P}|\mathcal{P}^{\mathbb{A}} \vee \mathcal{P}^-)$$

where $\mathbb{A} = (-\infty,\infty) \times (-\infty,-1]$, $\mathcal{P}^- = \mathcal{P}^{(-\infty,-1]\times\{0\}}$. *(The right hand side will be alternatively written as* $H(\mathcal{P}|\mathcal{P}^{\mathbb{A}'})$*, where* $\mathbb{A}' = \mathbb{A} \cup ((-\infty,-1] \times \{0\})$*.)*

Proof. All facts and formulas cited in this proof come from [1]. By Fact 2.3.4 (formula (2.3.5)), the right hand side equals $h(S,\mathcal{P}|\mathcal{P}^{\mathbb{A}})$. This, in turn, equals the limit of conditional dynamical entropies $h(S,\mathcal{P}|\mathcal{P}^{\{0\}\times[-n,-1]})$ (see Fact 2.4.16 (formula (2.4.17) and Fact 2.3.7 (formula (2.3.8))). Next, using Fact 2.4.2 (formula (2.4.4), this conditional entropy equals the difference $h(S,\mathcal{P}^{\{0\}\times[-n,0]}) - h(S,\mathcal{P}^{\{0\}\times[-n,-1]}) = h(S,\mathcal{P}^{\{0\}\times[-n,0]}) - h(S,\mathcal{P}^{\{0\}\times[-n+1,0]})$. Thus the sequence $h(S,\mathcal{P}^{\{0\}\times[-n+1,0]})$ has decreasing increments and by Fact 2.1.1 the limit of the increments equals the nonincreasing limit of the $n$ths, leading to

$$H(\mathcal{P},\mathcal{P}^{\mathbb{A}} \vee \mathcal{P}^-) = \lim_n \tfrac{1}{n} \downarrow h(S,\mathcal{P}^{\{0\}\times[0,n-1]})$$

Further,

$$h(S,\mathcal{P}^{\{0\}\times[0,n-1]}) = \lim_m \downarrow \tfrac{1}{m} H(\mathcal{P}^{[0,m-1]\times[0,n-1]}).$$

Iterated noincreasing limits commute and equal the double limit. Hence

$$\lim_n \downarrow \tfrac{1}{n} h(T,\mathcal{P}^{\{0\}\times[0,n-1]}) = \lim_{m,n} \tfrac{1}{mn} H(\mathcal{P}^{[0,m-1]\times[0,n-1]}) = h(S,T,\mathcal{P}).$$

□

Proof of Theorem 2.1. As before, only one inequality is nontrivial. Consider a finite set F and enumerate it lexicographically, so that for each element $(i,j) \in F$, the set of all its predecessors in F lies in $\mathbb{A}' + (i,j)$. The rest of the proof is identical as before: we write $H(\mathcal{P}^F)$ as the sum of $|F|$ conditional entropies (each time the condition depends on the coordinates in the set of predecessors). Thus $\frac{1}{|F|}H(\mathcal{P}^F)$ is not smaller than the smallest term, which in turn is not smaller than $H(\mathcal{P}|\mathcal{P}^{\mathbb{A}'})$. □

3. Amenable right-orderable groups

Theorem 2.1 extends to amenable groups which are *right orderable*, i.e., admit a linear right-invariant order. Note that every such group is torsion-free. It is known that an Abelian group is orderable if and only if it is torsion-free.

Theorem 3.1. *Suppose G is a countable, amenable and right-orderable group. Consider an action (X, Σ, μ, G) with a partition $\mathcal{P}$. Define, as before*

$$h^*(G,\mathcal{P}) = \inf\{\tfrac{1}{|F|}H(\mathcal{P}^F) : \ |F| < \infty\}. \tag{3.1}$$

Then $h^*(G,\mathcal{P}) = h(G,\mathcal{P})$, where the right hand side denotes the usual amenable entropy of the partition.

Proof. As before, we will focus on the nontrivial inequality. Let $F = \{g_1,\dots,g_k\}$ be an arbitrary finite set ordered increasingly. As in the proof of Theorem 1.2, there exists $i \in \{1,2,\dots,k\}$ such that

$$\tfrac{1}{|F|}H(\mathcal{P}^F) \ge H(\mathcal{P}^{\{g_i\}}|\mathcal{P}^{\{g_1,g_2,\dots,g_{i-1}\}}).$$

We will denote by E the set $\{g_1,g_2,\dots,g_{i-1}\}g_i^{-1}$. Let $\mathbf{F}$ be an element of the Følner sequence which is $(E,\frac{\epsilon}{|E|})$-invariant. Then $(1-\epsilon)$-percent of the elements g of $\mathbf{F}$ satisfy the condition $Eg \subset \mathbf{F}$ (the set of such elements will be denoted by $\mathbf{F}'$). Since the order is right-invariant, for $g \in \mathbf{F}'$ we have $Eg \subset \mathbf{F}_g$, where $\mathbf{F}_g$ denotes the set of predecessors of g contained in $\mathbf{F}$.

Now we can write

$$\begin{aligned}
&\tfrac{1}{|\mathbf{F}|}H(\mathcal{P}^{\mathbf{F}}) = \\
&\quad \tfrac{1}{|\mathbf{F}|}\sum_{g\in\mathbf{F}} H(\mathcal{P}^{\{g\}}|\mathcal{P}^{\mathbf{F}_g}) = \tfrac{1}{|\mathbf{F}|}\Big(\sum_{g\in\mathbf{F}'} H(\mathcal{P}^{\{g\}}|\mathcal{P}^{\mathbf{F}_g}) + \sum_{g\in\mathbf{F}\setminus\mathbf{F}'} H(\mathcal{P}^{\{g\}}|\mathcal{P}^{\mathbf{F}_g})\Big) \le \\
&\quad \tfrac{1}{|\mathbf{F}|}\sum_{g\in\mathbf{F}} H(\mathcal{P}^{\{g\}}|\mathcal{P}^{Eg}) + \epsilon\log\#\mathcal{P} = H(\mathcal{P}^{\{g_i\}}|\mathcal{P}^{\{g_1,g_2,\dots,g_{i-1}\}}) + \epsilon\log\#\mathcal{P} \le \\
&\qquad\qquad \tfrac{1}{|F|}H(\mathcal{P}^F) + \epsilon\log\#\mathcal{P}.
\end{aligned}$$

Since ϵ is arbitrarily small, this ends the proof. □

4. Finite cyclic groups

Theorem 3.1 applies to all torsion-free Abelian groups. The first step towards general Abelian groups is handling the case of finite cyclic groups $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ (where p is not necessarily a prime number).

Theorem 4.1. *If $G = \mathbb{Z}_p$ (for some $p \in \mathbb{N}$) then $h^*(G,\mathcal{P}) = h(G,\mathcal{P})$.*

Proof. Note that in this case $h(G,\mathcal{P}) = \frac{1}{|G|}H(\mathcal{P}^G)$ (this holds for any finite group). So, we only need to prove that for any finite set $F \subset G$ we have $\frac{1}{|F|}H(\mathcal{P}^F) \ge \frac{1}{|G|}H(\mathcal{P}^G)$. We will use the following easy observation: Let $\bar{f}$ denote the average value of a function $f : \mathbb{Z}_p \to \mathbb{R}$. Let $F \subset \mathbb{Z}_p$. Then there exists $k \in \mathbb{Z}_p$ such that the average value $\bar{f}_{F+k}$ of f over $F+k$ is not smaller than $\bar{f}$. We skip the elementary proof obtained by averaging over k.

Now we represent $\mathbb{Z}_p$ as $\{0,1,\dots,p-1\}$ with addition modulo p and we define $f : \mathbb{Z}_p \to \mathbb{R}$ by $f(0) = H(\mathcal{P}^{\{0\}})$, $f(1) = H(\mathcal{P}^{\{1\}}|\mathcal{P}^{\{0\}})$, $f(2) = H(\mathcal{P}^{\{2\}}|\mathcal{P}^{\{0,1\}}),\dots,$ $f(p-1) = H(\mathcal{P}^{\{p-1\}}|\mathcal{P}^{\{0,1,\dots,p-2\}})$. Clearly, we have $\bar{f} = h(G,\mathcal{P})$. Next, if $F = \{n_1,n_2,\dots,n_k\}$ (ordered increasingly), we write

$$H(\mathcal{P}^F) = H(\mathcal{P}^{\{n_1\}}) + H(\mathcal{P}^{\{n_2\}}|\mathcal{P}^{\{n_1\}}) + \cdots + H(\mathcal{P}^{\{n_k\}}|\mathcal{P}^{\{n_1,n_2,\dots,n_{k-1}\}}).$$

Note that the ith term in this sum is not smaller than $f(n_i)$, thus $\frac{1}{|F|}H(\mathcal{P}^F)$ is not smaller than the average $\bar{f}_F$ of f over F. But, by invariance of the measure, the expression $\frac{1}{|F|}H(\mathcal{P}^F)$ will not change, if F is replaced by $F+k$ (for any $k \in \mathbb{Z}_p$), which implies that $\frac{1}{|F|}H(\mathcal{P}^F)$ is not smaller than any average of the form $\bar{f}_{F+k}$, and hence also not smaller than $\bar{f} = h(G,\mathcal{P})$. □

5. Amenable groups – the general case

The experiments with ordered amenable groups and finite cyclic groups have led us to discovering a method allowing to handle general amenable groups.

Theorem 5.1. *Let G be a countable amenable group. Consider an action (X, Σ, μ, G) and a partition $\mathcal{P}$. Then $h^*(G, \mathcal{P}) = h(G, \mathcal{P})$.*

Proof. As in the proof of Theorem 3.1, we only need to show that for every finite set $F \subset G$ and every $\epsilon > 0$ there exists a set $\mathbf{F}$ in the Følner sequence, such that $\frac{1}{|\mathbf{F}|}H(\mathcal{P}^{\mathbf{F}}) \le \frac{1}{|F|}H(\mathcal{P}^F) + \epsilon C$ (where C is some constant). It suffices to consider sets F containing the unity of G.

Let $\mathbf{F}$ be an element of the Følner sequence which is $(F \cup F^{-1}, \frac{\delta}{|F|})$-invariant ($\delta$, depending only on ϵ and $|F|$, will be specified later). Given an order τ of $\mathbf{F}$ we denote by $\mathbf{F}^\tau_g$ the set of predecessors of $g \in \mathbf{F}$. Then

$$H(\mathcal{P}^{\mathbf{F}}) = \sum_{g \in \mathbf{F}} H(\mathcal{P}^{\{g\}} | \mathcal{P}^{\mathbf{F}^\tau_g}) =: \sum_{g \in \mathbf{F}} A(\tau, g).$$

Likewise, given an order σ of $|F|$ (and using an analogous notation F^σ_g) we have

$$H(\mathcal{P}^F) = \sum_{g \in F} H(\mathcal{P}^{\{g\}} | \mathcal{P}^{F^\sigma_g}) =: \sum_{g \in F} A'(\sigma, g).$$

We know that $(1-\delta)$-percent of the elements g of $\mathbf{F}$ satisfy the condition $Fg \subset \mathbf{F}$. We denote the set of such elements g by $\mathbf{F}'$. We will need to consider all possible positions of the shifted set F inside $\mathbf{F}$. Such positions are indexed by the elements of $\mathbf{F}'$ and we will use the letter f' (rather than g) to denote them. Every pair (τ, f') determines the positions of all elements of the set Ff' in the ordered set $\mathbf{F}$, and hence induces an order $\sigma_{\tau,f'}$ on F (τ introduces an order in Ff', which then transports to F via the multiplication by ${f'}^{-1}$).

Let us define $A(f', \tau, g) = A(\tau, g)$ (i.e., we only add the index f'). We have produced a three-dimensional matrix and we will analyze its one-dimensional *rows* (here f' and g are fixed while the permutation τ varies), *columns* (the shifting f' of F varies), and *stacks* (the position g in the sum varies). We know that the sum of each stack equals $H(\mathcal{P}^{\mathbf{F}})$, so the total sum of the matrix is $|\mathbf{F}|!|\mathbf{F}'|H(\mathcal{P}^{\mathbf{F}})$. Now, in this matrix we "mark" the elements $A(f', \tau, g)$ for which $g \in Ff'$, i.e., representing the (conditional) entropy of a coordinate belonging to the shifted copy of our small set F. We will focus on summing only the marked terms.

Notice that given τ and g, the term $A(\tau, g)$ appears as a marked $A(f', \tau, g)$ in the corresponding column exactly $|F|$ times, except when there is an $f \in F$ such that ff' never equals g, i.e., when $F^{-1}g$ is not fully contained in $\mathbf{F}'$. This is possible for only a small percentage of g's in $\mathbf{F}$, the percentage γ depending on δ and $|F|$. We denote the corresponding small subset of $\mathbf{F}$ by B. So, the sum of all marked terms in our matrix ranges between $|\mathbf{F}|!|F|H(\mathcal{P}^{\mathbf{F}})$ and $|\mathbf{F}|!|F|(H(\mathcal{P}^{\mathbf{F}}) - \gamma|\mathbf{F}|\log \#\mathcal{P})$ (due to the missing terms corresponding to $g \in B$).

At this point we replace each marked term $A(f', \tau, g) = H(\mathcal{P}^g | \mathcal{P}^{\mathbf{F}^\tau_g})$ by a not smaller term $A''(f', \tau, g) = H(\mathcal{P}^g | \mathcal{P}^{F{f'}^\tau_g})$, where $F{f'}^\tau_g$ denotes the set of the τ-predecessors of g belonging to Ff'. The sum of the new marked terms has not dropped, while every new term equals $A'(\sigma_{\tau,f'}, g{f'}^{-1})$, a term appearing in the sum representing the entropy $H(\mathcal{P}^F)$ developed in accordance to the order $\sigma_{\tau,f'}$ on F. Our task is to show that every term of the form $A'(\sigma, g)$ ($g \in F$, σ is an order of F) appears in the sum of the marked terms the same number of times. Fix an $f' \in \mathbf{F}'$ and observe the corresponding two-dimensional slice of the matrix. It is clear that the orders τ of $\mathbf{F}$ induce orders σ of Ff' in equal proportions, i.e., every σ is obtained from the same number of τ's (the number being $\frac{|\mathbf{F}|!}{|F|!}$). Once σ is established, every $g \in Ff'$ determines both a marked element of the slice and a term $A'(\sigma, g)$. Thus, among all marked elements in the slice we will find every term $A'(\sigma, g)$ precisely $\frac{|\mathbf{F}|!}{|F|!}$ times. Hence the sum of the (new) marked terms in a slice equals $|F|!\frac{|\mathbf{F}|!}{|F|!}H(\mathcal{P}^F) = |\mathbf{F}|!H(\mathcal{P}^F)$, and in the whole matrix it amounts to $|\mathbf{F}|!|\mathbf{F}'|H(\mathcal{P}^F)$. We have proved that

$$|\mathbf{F}|!|\mathbf{F}'|H(\mathcal{P}^F) \geq |\mathbf{F}|!|F|(H(\mathcal{P}^{\mathbf{F}}) - \gamma|\mathbf{F}|\log\#\mathcal{P}),$$

i.e., $\frac{1}{|F|}H(\mathcal{P}^F) \geq \frac{1}{|\mathbf{F}'|}(H(\mathcal{P}^{\mathbf{F}}) - \gamma|\mathbf{F}|\log\#\mathcal{P}) \geq \frac{1}{|\mathbf{F}|}H(\mathcal{P}^{\mathbf{F}}) - \gamma\log\#\mathcal{P}$. Since γ is arbitrarily small, the proof is completed. □

6. Beyond amenability

The "mindblowingly" simple formula $h^*(G, \mathcal{P})$ can be applied to processes under actions of any countable groups. (It can be applied to uncountable groups as well, however, it will typically yield zero; such is the case of flows.) How good is this formula for countable non-amenable groups. The answer depends on the properties we expect from a good notion of dynamical entropy.

The notion $h^*(G, \mathcal{P})$ has the following advantages:

- It is completely universal, can be defined for arbitrary countable gropus.
- It is extremely simple, requires no details of the group (for instance in amenable groups it is formulated without referring to any Følner sequence).
- It has a very convincing interpretation (entropy is lost only in finite-dimensional dependencies and all such losses matter).
- Bernoulli shifts have "full" entropy (equal to the static entropy of the partition).

Disadvantages can be detected by examining the action of the free group F_2 on two generators, and they include

- It can increase under factors.
- It can change with change of a generator.

Example 6.1. Let F_2 denote the free group with two generators a and b, and consider $X = \{-1, 1\}^{F_2}$ with the shift action, the Bernoulli $(\frac{1}{2}, \frac{1}{2})$-measure, and the zero-coordinate partition $\mathcal{P} = \{[-1], [1]\}$. Clearly, $H(\mathcal{P}) = \log 2$ and $h^*(F_2, \mathcal{P}) = \log 2$. Next, consider the function $\psi : X \to \{-1, 1\} \times \{-1, 1\}$ given by

$$\psi(x) = (x(\phi)x(a), x(\phi)x(b))$$

and the associated four-element partition $\mathcal{R}$ (we have $\mathcal{R} \preccurlyeq \mathcal{Q}$). It is not hard to see that the process generated by $\mathcal{R}$ is the $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$-Bernoulli shift: the one-dimensional distributions are independent. So, $H(\mathcal{R}) = \log 4$ and so equals $h^*(F_2, \mathcal{R})$. On the other hand, the process generated by $\mathcal{R}$ is clearly a factor of that generated by $\mathcal{P}$. This failure of the "factors condition" cannot be avoided by any entropy notion satisfying the "Bernoulli shifts" condition.

Now let $E = \{\phi, a, b\} \subset F_2$ and consider $\mathcal{Q} = \mathcal{P}^E$. Clearly, this partition is another generator of the process generated by $\mathcal{P}$ (the generated processes are isomorphic). For any finite set $F \subset F_2$ we have $H(\mathcal{Q}^F) = H(\mathcal{P}^{EF}) = |EF|\log 2$.

However, the ratio $\frac{|FE|}{|F|}$ does not drop below 2 (and can be arbitrarily close to 2). Hence $h^*(F^2,\mathcal{Q}) = 2h^*(F^2,\mathcal{P}) = \log 4$. Sofic entropy behaves better in this aspect, so that it can be defined for measure-preserving actions regardless of the generator, and becomes an isomorphism invariant.

So, either we accept $h^*(G,\mathcal{P})$ as a parameter associated with a concrete *process*, maintaining its simplicity and interpretation, or we try to force it to become an isomorphism invariant. As an attempt in this direction we propose two invariants, both equal to $h^*(G,\mathcal{P})$ for actions of amenable groups. Unfortunately, we are unable to verify whether these new notions fulfill the Bernoulli shift condition in the general case.

Definition 6.2.

$$h^{**}(X,\Sigma,\mu,G) = \inf\{h^*(G,\mathcal{P}):\ \mathcal{P} \textit{ is a generator}\}.$$

$$h^{***}(X,\Sigma,\mu,G) = \inf\{H(\mathcal{P}):\ \mathcal{P} \textit{ is a generator}\}.$$

Note that the latter notion has nothing to do with $h^*(G,\mathcal{P})$, we were driven to it just by analogy to $h^{**}(X,\Sigma,\mu,G)$. Its validity (i.e., the fact that it equals the dynamic entropy) for actions of amenable groups follows easily from Sinai's theorem and the fact that every factor has an ϵ-independent complementary factor (see e.g. Corollary 4.4.8 in [1] for $\mathbb{Z}$-actions). Could it be another useful more general notion?

This is where we stop for now.

References

1. T. Downarowicz: **Entropy in dynamical systems**, Cambridge University Press, *New Mathematical Monographs 18*, Cambridge 2011