Families of Dirac operators and quantum affine groups

Deforming twisted K-theory

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References

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Outline of the talk

- Background: Twisted K-theory from Dirac type operators on loop groups
- q-Deformation of the Dirac family
- The q-fermionic algebra and generalized affine Hecke algebra
- Quantum adjoint module
- Twisting and the central element in $U_q(\hat{\mathfrak{g}})$

Gerbes and Fredholm operators

X is a topological parameter space, $Fred_*$ the space of self-adjoint Fredholm operators in a complex Hilbert space H with both positive and negative essential spectrum. This is a universal classifying space for K^1. Actually, one can take as the **definition:**

$$K^1(X) = \{\text{homotopy classes of maps } f : X \to Fred_*\}$$

Without loss of generality we can require that the Fredholm operators have a discrete spectrum. In the even case

$$K^0(X) = \{\text{homotopy classes of maps } f : X \to Fred\}$$

where $Fred$ is the space of all Fredholm operators in H.

The Dixmier-Douady class

The Chern character

$$ch : K^1(X) \to H^{odd}(X, \mathbf{Z})$$

is an additive map to odd cohomology classes. In particular, the degree 3 component $DD(f) = ch_3(f)$ of $[f] \in K^1(X)$ is called the **Dixmier-Douady class** of the gerbe defined by the the family $f(x)$ of Fredholm operators. In the de Rham cohomology an equivalent construction of $DD(f)$ comes from the family $L_{\lambda\lambda'}$ of complex line bundles. One can choose the curvature forms $\omega_{\lambda\lambda'}$ such that

$$\omega_{\lambda\lambda'} + \omega_{\lambda'\lambda''} = \omega_{\lambda\lambda''}$$

and with a partition of unity $\sum \rho_\lambda = 1$ subordinate to the cover by the open sets U_λ one has

$$DD(f) = \sum_\lambda d\rho_\lambda \wedge \omega_{\lambda\lambda'}$$

and this does not depend on the choice of λ'.

Twisting K-theory with the D-D class

The previous example can generalized: Let $P \to X$ be a principal bundle with a right action of a group $\mathcal{G}$. Fix a cocycle

$$\omega : P \times \mathcal{G} \to PU(H), \text{ with } \theta(p; g_1 g_2) = \theta(p; g_1)\theta(pg_1; g_2)$$

where $PU(H) = U(H)/S^1$. Then a map $f : P \to Fred(H)$ with

$$f(pg) = \theta(p; g)^{-1} f(p)\theta(p; g)$$

defines an element in the twisted K-group $K^0(X; \theta)$. The group K^1 is defined similarly using $Fred_*$ instead of $Fred$.
The groups $K^*(X, \theta)$ actually depend only on the class of the $PU(H)$ bundle defined by the cocycle θ. This class is the Dixmier-Douady class in $H^3(X, \mathbf{Z})$.

Example: The WZW model

Families of Dirac operators D_A transform covariantly under the (projective) gauge group action, defining an element in $K^*(\mathcal{A}/\mathcal{G}, \theta)$, where θ is defined by the projective action $g \mapsto \hat{g}$ in the Fock spaces? **False:** The quantized Dirac operators are essentially positive, we need operators with both negative and positive essential spectrum. Solution: Hamiltonians in supersymmetric WZW model:

$$
\begin{aligned}
Q_A &= i\psi_a^n T_a^{-n} + \frac{i}{12}\lambda_{abc}\psi_a^n\psi_b^m\psi_c^{-n-m} + i(k+\kappa)\psi_a^n A_a^{-n} \\
\psi_a^n\psi_b^m &+ \psi_b^m\psi_a^n = 2\delta_{ab}\delta_{n,-m} \\
[T_a^n, T_b^m] &= \lambda_{abc}T_c^{n+m} + k\delta_{ab}n\delta_{n,-m}.
\end{aligned}
$$

Here A_a^n's are the Fourier components of a vector potential on the circle.

The WZW model

The family Q_A transforms covariantly under the projective representation of level $k+\kappa$ the loop group $\mathcal{G} = LG$ defining an element in $K(G, k+\kappa)$ corresponding to the D-D class $[H]$ in $H^3(G, \mathbf{Z})$ equal to $k+\kappa$ times the basic class in $H^3(G) = \mathbf{Z}$ when G is a simple simply connected compact Lie group. Actually, since $\mathcal{A}/\Omega G = G$ and $G \subset LG$, we have an G equivariant class, element of $K^*_G(G, H)$.

Morally, the family Q_A is a family of Dirac operators on the loop group LG, coupled to a gauge connection A on a complex line bundle over LG.

Quantum affine algebra

$\mathfrak{g}$ a simple finite-dimensional Lie algebra, $\hat{\mathfrak{g}}$ the associated affine Lie algebra. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is generated by $e_0, e_1, \dots, e_\ell, f_0, f_1, \dots, f_\ell, K_0, K_1, \dots, K_\ell, K_0^{-1}, \dots, K_\ell^{-1}$ with the relations

$$\begin{aligned}
[e_i, f_i] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, K_i K_j = K_j K_i \\
K_i e_j K_i^{-1} &= q^{\alpha_{ij}} e_j, K_i f_j K_i^{-1} = q^{-\alpha_{ij}} f_j \\
\sum_{k=0}^{1-a_{ij}} & (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_q e_i^{1-a_{ij}-k} e_j e_i^k = 0 (i \neq j) \\
\sum_{k=0}^{1-a_{ij}} & (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_q f_i^{1-a_{ij}-k} f_j f_i^k = 0 (i \neq j)
\end{aligned}$$

with

$$\left[\begin{array}{c} m \\ k \end{array}\right]_q = \frac{m_q(m-1)_q \dots (m-k+1)_q}{k_q(k-1)_q \dots 1_q}$$

$$k_q = 1 + q + \dots q^{k-1}$$

q is a positive real number in this talk and the integers a_{ij} are the matrix elements of the Cartan matrix of $\hat{\mathfrak{g}}$.

The Dirac operator

Let A_i^n with $n \in \mathbf{Z}$ and $i = 0, 1, \ldots dim\, \mathfrak{g}$ be a basis for the q-affine adjoint module. Under $\mathfrak{g}$ each 'Fourier mode' A^n transforms acording to the adjoint representation of $U_q(\mathfrak{g})$, which is a q-deformation of the adjoint representation of $\mathfrak{g}$. The generator e_0 increases the index n by one unit, f_0 decreases it by one unit. For example, for $\mathfrak{g} = \mathfrak{sl}(2)$ one has the explicit formulas

$$
\begin{aligned}
e_1 A_1^n &= f_0 A_1^n = 0, f_1 A_1^n = A_0^n, e_0 A_1^n = A_0^{n+1} \\
e_1 A_0^n &= (q + q^{-1}) A_1^n, f_0 A_0^n = (q + q^{-1}) A_1^{n-1} \\
f_1 A_0^n &= A_{-1}^n, e_0 A_0^n = A_{-1}^{n+1} \\
e_1 A_{-1}^n &= (q + q^{-1}) A_0^n, f_0 A_{-1}^n = (q + q^{-1}) A_0^{n-1}, f_1 A_{-1}^n = 0 = e_0 A_{-1}^n \\
K_1 A_i^n &= q^{2i} A_i^n = K_0^{-1} A_i^n.
\end{aligned}
$$

The Dirac operator

The vectors A_i^n will be constructed as operators acting in a Fock space carrying a representation of $U_q(\hat{\mathfrak{g}})$ such that the adjoint action is given by

$$x.A_i^n = \sum_{(x)} x' A_i^n S(x'') \text{ for } x \in U_q(\hat{\mathfrak{g}}),$$

where $S : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$ is the antipode and $\Delta(x) = \sum_{(x)} x' \otimes x''$ is the coproduct $\Delta : U_q \to U_q \otimes U_q$. We also need the Clifford algebra generated by elements ψ_i^n acting in the Fock space and transforming under $U_q(\hat{\mathfrak{g}})$ according to the dual adjoint representation (which in fact is equivalent to the adjoint representation).

The Dirac operator

The Dirac operator Q is acting in $H_f \otimes H_b$ where H_f is the q-fermionic Fock space and H_b carries another highest weight representation of $U_q(\hat{\mathfrak{g}})$.

$$Q = \sum \psi_i^n \otimes B_i^{-n} + \frac{1}{3} \sum \psi_i^n A_i^{-n} \otimes 1$$

where B_i^n is another copy of the adjoint module, acting in the space H_b.

The adjoint module

Let R be the universal R-matrix for the algebra U_q. An explicit construction is given in [KT]. Following [DG], we can then define a basis for vectors in a submodule $A \subset U_q$ transforming according to an adjoint representation

$$ad_q(x)v = \sum_{(x)} x' v S(x'')$$

of U_q on itself. A basis is defined as

$$A_i^n = \sum K_{n,i}^{m,\alpha;p,\beta} (\pi_{m,\alpha;p,\beta} \otimes id) A,$$

where $A = (R^T R - 1)/h$, with $e^h = q$ and $R^T = \sigma R \sigma$, where σ permutes the factors in the tensor product $U_q \otimes U_q$. Here $\pi_{m,\alpha;p,\beta}$ are the matrix elements in the defining representation V of U_q.

The adjoint module

For example, for $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$ the basis in the defining representation is v_i^n with $n \in \mathbf{Z}$ and $i = -1, 0, 1$ and $\alpha, \beta = \pm$. The numerical coefficients K come from the identification of the basis of the adjoint representation as linear combinations of the basis vectors in $V \otimes V$.
The action of the Serre generators in the defining representation is

$$\begin{aligned} e_1 v_+^n &= f_0 v_+^n = 0, f_1 v_+^n = v_-^n, e_0 v_+^n = v_-^{n+1}, e_0 v_+^n = v_-^{n-1} \\ e_1 v_-^n &= v_+^n,\ f_0 v_-^n = v_+^{n-1}, e_0 v_-^n = 0 = f_1 v_-^n \\ K_1 v_\pm^n &= q^{\pm 1} v_\pm^n = K_0^{-1} v_\pm^n. \end{aligned}$$

The affine Hecke algebra for $\hat{\mathfrak{g}}$ is defined through the relations [Leclerc] coming from the R-matrix $\check{\mathrm{R}} = \sigma R$ in the tensor product $V^0 \otimes V^0$. The matrix satisfies

$$(\check{\mathrm{R}} - q^{-1})(\check{\mathrm{R}} + q) = 0,$$

since $-q$ and q^{-1} are the only eigenvalues of the invertible matrix $\check{\mathrm{R}}$. Denote by Y_1 the shift operator which sends $v_i^n \otimes v_j^m$ to $v_i^{n+1} \otimes v_j^m$ and by Y_2 the corresponding shift operator acting on the second tensor factor. The matrix $\check{\mathrm{R}}$ acting on V is then defined using the relations

$$\check{\mathrm{R}} Y_1 = Y_2 \check{\mathrm{R}}^{-1}, \; \check{\mathrm{R}} Y_2 = Y_1 \check{\mathrm{R}} + (q - q^{-1}) Y_2.$$

Actually, the second relation follows from the first and the minimal polynomial relation.

Generalized affine Hecke algebra

Now the braiding relations are given by setting the ideal in the tensor algebra of V generated by the elements

$$(q^{-1} + \check{R})(V \otimes V)$$

equal to zero. These have in particular the consequence that any $v_i^n v_j^m$ with $n > m$ can be written as a linear combination of vectors $v_k^p v_l^q$ with $p + q = n + m$ and $p \leq q$. In the zero mode space V^0 the meaning of the braiding relations is that they project out the 'symmetric' part of the tensor product $V^0 \otimes V^0$. The 3-dimensional representation is the eigenspace of $\check{R}$ with eigenvalue q^{-1} and the 1-dimensional component corresponds to the eigenvalue $-q$.

Generalized affine Hecke algebra

To complete the construction of the Dirac operator we need also the generalized Clifford algebra in the coadjoint representation. The algebra is generated by vectors ψ_i^n with $n \in \mathbf{Z}$ and $i = 1, 0, -1$. The defining relations are given by braiding relations and an invariant (nonsymmetric) bilinear form. The braiding relations are defined recursively like in the case of V, V^*, with the difference that since the R-matrix Ř in the adjoint representation has 3 instead of 2 different eigenvalues, which are now $-q^{-2}, q^2, q^{-4}$, with multiplicities $3, 5, 1$ respectively.

Generalized Hecke algebra

The negative eigenvalue corresponds again to a 3-dimensional 'antisymmetric' representation and the positive eigenvalues to a 6-dimensional 'symmetric' representation; the latter contains the 1-dimensional trivial representation.
The Hecke algebra is replaced by a generalized Hecke algebra,

$$
\begin{aligned}
&Y_1 Y_2 = Y_2 Y_1 \\
&(\check{\mathrm{R}} - q^2)(\check{\mathrm{R}} - q^{-4})(\check{\mathrm{R}} + q^{-2}) = 0 \\
&\check{\mathrm{R}} Y_1 = Y_2 \check{\mathrm{R}}^{-1},\ \check{\mathrm{R}} Y_2 = Y_i \check{\mathrm{R}} + (q^2 - q^{-2}) Y_2
\end{aligned}
$$

where the middle relation is the minimal polynomial of the diagonalizable matrix $\check{\mathrm{R}}$.

q-Clifford algebra

The generalized symmetric tensors correspond to positive eigenvalues of Ř. In the Clifford algebra symmetrized products are identified as scalars times the unit. That is, we fix a $U_q(\widehat{\mathfrak{sl}}(2))$ invariant bilinear form B and the Clifford algebra is defined as the tensor algebra over V modulo the ideal generated by

$$P(u \otimes v) - 2B(u, v) \cdot 1$$

where P is the projection on positive spectral subspace of Ř. In the case when V is the adjoint module for $U_q(\widehat{\mathfrak{sl}}(2))$ one can fix B by identifying the first factor V as the dual V^* and using the natural pairing $V^* \otimes V \to \mathbf{C}$. Alternatively, one can view B as the projection onto the 1-dimensional trivial submodule inside of the 'symmetric module'.

The action of $U_q(\widehat{\mathfrak{sl}}(2))$ on Q

In the nondeformed case one has for an infinitesimal gauge transformation $X \in L\mathfrak{g}$

$$[X, Q] = (k+\kappa)\sum(-n)\psi_i^n X_i^{-n} = (k+\kappa) < \psi, dX >$$

and for a family of operators $Q_A = Q + (k+\kappa)\psi_i^n A_i^{-n}$

$$[X, Q_A] = (k+\kappa) < \psi, [A, X] + dX > .$$

In q-deformed case A is to be understood as a vector in the adjoint module extended by $\mathbf{C}c$. Thus

$$x._c v = x.v + \lambda(x)c$$

with λ a linear form on $U_q(\widehat{\mathfrak{sl}}(2))$.