PLÜNNECKE INEQUALITIES FOR COUNTABLE ABELIAN GROUPS

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1. Plünnecke inequalities for finite sets

Let G be a countable abelian group, $A, B \subset G$ finite sets. We define

$$A + B = \{a + b \,|a \in A, b \in B\},$$

and for every $k \geq 1$:

$$kA = \{a_1 + \ldots + a_k \,|\, a_1, \ldots, a_k \in A\}.$$

Natural Question: Is it true that if $|2A|$ is small then $|3A|$ is small? More precisely, if $|A + A| \leq C|A|$ then $|A + A + A| = O_C(|A|)$?

Answer: Yes. Moreover, we have

$$|A + A + A| \leq C^3|A|.$$

Method: Plünnecke inequalities.
We define for every $k \geq 1$, k-th magnification ratio:

$$m_k = \min_{\emptyset \neq A' \subset A} \frac{|A' + kB|}{|A'|}.$$

Theorem (Plünnecke 1970) $m_k^{1/k}$ is non-increasing, i.e., we have

$$m_1 \geq m_2^{1/2} \geq m_3^{1/3} \geq \ldots \geq m_k^{1/k} \geq \ldots$$

Application: We apply the inequality $m_3 \leq m_1^3$ in the case $A = B$, and we get:

$$\text{if } |A + A| \leq C|A| \text{ then } |A + A + A| \leq C^3|A|.$$

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2. Plünnecke inequalities for infinite sets

For $A \subset \mathbb{N} = \{0, 1, 2, \ldots\}$ (not necessarily finite) the Shnirel'man density is defined by

$$\sigma(A) = \inf_{n \geq 1} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.$$

Based on the finite inequalities (in a non-trivial way) it follows:

Theorem (Plünnecke 1970): If $A, B \subset \mathbb{N}$, and $0 \in B$ then

$$\sigma(A + B) \geq \sigma(A)^{1-1/k} \sigma(kB)^{1/k}.$$

Corollary: If $0 \in B \subset \mathbb{N}$ satisfies that $kB = \mathbb{N}$ then $\sigma(A + B) \geq \sigma(A)^{1-1/k}$.

Remark: It is enough for the last corollary that the lower asymptotic density of kB to be equal one. For instance, it follows that for the set P of all primes and zero, we have $\sigma(A + P) \geq \sigma(A)^{3/4}$.

Recently, Plünnecke inequality for Shnirel'man's density was extended to *upper Banach density.*

Theorem (Renling Jin 2011): For $A, B \subset \mathbb{N}$ we have

$$d^*(A + B) \geq d^*(A)^{1-1/k} d^*(kB)^{1/k},$$

where

$$d^*(A) = sup\{d(A) \,|\, \text{d is inv. density}\} = \limsup_{b_n - a_n \to \infty} \frac{|A \cap [a_n, b_n]|}{b_n - a_n + 1}.$$

3. Furstenberg correspondence for sumsets

For G a countable amenable group (every group for which there exists an invariant density; all abelian groups are amenable), for any set $A \subset G$ we can correspond a compact metric space X, with action of G by homeomorphisms, an **ergodic** G-invariant Borel probability measure on X, and a measurable set $\tilde{A} \subset X$ such that:

For every $B \subset G$ we have $d^*(B + A) \geq \mu \left(\bigcup_{g \in B} g\tilde{A} \right)$, and $\mu(\tilde{A}) = d^*(A)$.

Remark: The correspondence principle for sumsets was first noticed by Beiglbock, Bergelson and Fish in 2010. It is analogous to Furstenberg's correspondence principle for intersections which was used to prove that multiple recurrence implies Szemeredi's theorem. This form of the correspondence principle (B is not necessarily a finite set) was noticed by Björklund and Fish.

4. Main results

Let G be a countable abelian group.

Definition (Björklund, Fish) Let (X, μ) be a measure-preserving G-system, let $B \subset G$ (not necessarily finite set), $A \subset X$ a measurable set of positive measure. For any $k \geq 1$, the k-th magnification ratio is

$$M_k = \inf \left\{ \frac{\mu\left(\bigcup_{g \in kB} gA'\right)}{\mu(A')} \mid A' \subset A, \mu(A') > 0 \right\}.$$

Main Theorem (Björklund, Bulinski, Fish). The sequence $M_k^{1/k}$ is non-increasing, i.e., $M_1 \geq M_2^{1/2} \geq M_3^{1/3} \geq \ldots \geq M_k^{1/k} \ldots$.

Remark. The inequality $M_1 \geq M_k^{1/k}$ was proved by Björklunf and Fish. The general case was done by Bulinski and Fish.

Corollary 1. Suppose G is a countable abelian group, $A, B \subset G$ (infinite). Then for integers $1 \leq j \leq k$ we have

$$d^*(A + jB) \geq d^*(A)^{1-j/k} d^*(kB)^{j/k}.$$

Definition (ergodic set). A set $B \subset G$ is **ergodic** if for every ergodic G-measure-preserving system , and any set $A \subset X$ of positive measure we have

$$\mu(\bigcup_{g \in B} gA) = 1.$$

By use of the following quite easy fact:

Lemma. Let G be a countable amenable group, and let $B \subset G$ be of positive density, i.e., $d^*(B) > 0$. Then for every ergodic G-system (X, μ), and every measurable $A \subset X$ we have

$$\mu(\bigcup_{g \in B} gA) \geq d^*(B).$$

we obtain

Corollary 2. Suppose G is an abelian group, and $B \subset G$ satisfies that kB is ergodic, then for every $1 \leq j < k$ we have

- For every $A \subset G$ we have $d^*(A + jB) \geq d^*(A)^{1-j/k}$.
- For every ergodic G-system (X, μ), and every $A \subset X$ of positive measure we have $\mu(\bigcup_{g \in jB} gA) \geq \mu(A)^{1-j/k}$.

5. Main ingredients in the proof of Main Theorem

There are two main ingredients in the proof:

- A. To prove Main Theorem in the case of a finite set of translates.
- B. To "push" the result to the infinite set of translates.

Let us discuss the main steps in every one of the ingredients.

A. Proof of Main Theorem for a finite set of translates (B is finite).

We use the recent works of Petridis.

The case $j = 1$: We adapt the following elegant lemma to our setting (including the proof).

Lemma (Petridis). Let G be any group, and let $A, B \subset G$ be finite sets. Let $\emptyset \neq X \subset A$ be such that

$$\alpha = \frac{|XB|}{|X|} \leq \frac{|ZB|}{|Z|}, \text{for every } Z \subset A.$$

Then for any finite set $C \subset G$ we have

$$|CXB| \leq \alpha |CX|.$$

Proof is by induction on $|C|$!!!

As an application of adapted Petridis's lemma to dynamical setting, we obtain that for a finite set $B \subset G$ we have $M_1^k \geq M_k$.

Another corollary of Petridis's lemma is the statement: $m_1^k \geq m_k$.

General case. It is based on the extension to the dynamical setting (Bulinski, Fish) of another work of Petridis – a new graph theoretic proof of Plünnecke inequalities. The Petridis's approach simplifies the Ruzsa-Plünnecke classical method.

We prove that if $B \subset G$ is finite, then the sequence $M_k^{1/k}$ is non-increasing.

B. "Pushing" the statement A. to the infinite set of translates.

The whole machinery for pushing argument has been done in the joint work with Björklund. The idea is to use a certain compactness argument. For doing so, for every $\delta > 0$ and every $k \geq 1$ we define the weighted magnification ratio $M_{k,\delta}$ to be

$$M_{k,\delta} = \inf \left\{ \frac{\mu\left(\bigcup_{g \in kB} gA'\right)}{\mu(A')} \mid A' \subset A, \mu(A') > \delta\mu(A) \right\}.$$

It is easy to see that it follows from statement A., that for every finite set $B \subset G$, and any $1 \leq j < k$ we have:

$$M_{k,\delta}^{1/k} \leq (1-\delta)^{-1/j} \left(\frac{\mu(\bigcup_{g \in jB} gA)}{\mu(A)} \right)^{1/j}.$$

The main Key statement is the following minimax result.

Proposition (Björklund, Fish). For any countable group G (not necessarily amenable), for any measure-preserving G-system, any measurable set A of positive measure, and any set $B \subset G$ we have

$$\inf \left\{ \frac{\mu\left(\bigcup_{g \in B} gA'\right)}{\mu(A')} \mid A' \subset A, \mu(A') > 0 \right\} \leq$$

$$\sup_{B' \subset B \text{ finite}} \inf \left\{ \frac{\mu\left(\bigcup_{g \in B'} gA'\right)}{\mu(A')} \mid A' \subset A, \mu(A') > \delta\mu(A) \right\}.$$