# Amorphic complexity

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We introduce amorphic complexity as a new topological invariant that measures the complexity of dynamical systems in the regime of zero entropy. Its main purpose is to detect the very onset of disorder in the asymptotic behaviour and it gives, for example, positive value to Denjoy examples on the circle and Sturmian subshifts, while being zero for all isometries and Morse-Smale-systems.

After discussing basic properties and examples, we show that amorphic complexity and the underlying asymptotic separation numbers can be used to distinguish almost automorphic minimal systems from equicontinuous ones. For symbolic systems, amorphic complexity equals the box dimension of the associated Besicovitch space. In this context, we concentrate on regular Toeplitz flows and give a detailed description of the relation to the scaling behaviour of the densities of the $p$-skeletons. Finally, we take a look at strange non-chaotic attractors appearing in so-called pinched skew-product systems. Continuous-time systems, more general group actions and the application to cut and project quasicrystals will be treated in subsequent work.

The paradigm example of a topological complexity invariant for dynamical systems is topological entropy, which measures the exponential growth, in time, of orbits distinguishable with finite precision. It can be used to compare the complexity of dynamical systems defined on arbitrary metric spaces and is central to the powerful machinery of thermodynamic formalism. There are, however, two situations where entropy does not provide very much information, namely when it either zero or infinite. In the latter case, mean topological dimension has been identified as a suitable substitute. Its theoretical significance is demonstrated, for example, by the fact that zero mean dimension is one of the few dynamical consequences of unique ergodicity [LW00].

In this talk, we introduce introduce amorphic complexity as a new topological invariant that measures the complexity of dynamical systems in the regime of

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zero entropy. One of its main purposes is to detect the very onset of dynamical complexity and the break of equicontinuity. In particular, it satisfies the following basic requirements.

(i) is an invariant of topological conjugacy and has other 'good properties';

(ii) gives value zero to isometries and Morse-Smale-systems;

(iii) is able to detect, as test cases, the complexity inherent in the dynamics of Sturmian shifts or Denjoy homeomorphisms on the circle, by taking positive values for such systems.

The standard approach to measure the complexity of zero entropy systems is to consider subexponential growth rates of distinguishable orbits, instead of exponentional ones, leading to the notions of power entropy and modified power entropy [HK02]. However, it turns out that power entropy gives positive values to Morse-Smale-systems, whereas modified power entropy is too coarse to distinguish Sturmian subshifts or Denjoy examples from irrational rotations. We are thus taking an alternative path, which leads us to define the notions of *asymptotic separation numbers* and *amorphic complexity*. In order to fix ideas, we concentrate on the dynamics of maps defined on metric spaces.

Let $(X, d)$ be a metric space and $f : X \to X$. Given $x, y \in X$, $\delta > 0$, $\nu \in (0, 1]$ and $n \in \mathbb{N}$ we let

$$S_n(f, \delta, x, y) \;=\; \#\left\{0 \le k < n \mid d(f^k(x), f^k(y)) \ge \delta\right\} . \tag{1}$$

We say that $x$ and $y$ are $(f, \delta, \nu)$*-separated with respect to* $f$ if

$$\varlimsup_{n\to\infty} \frac{S_n(f, \delta, x, y)}{n} \;\ge\; \nu \; . \tag{2}$$

A subset $S \subseteq X$ is said to be $(f, \delta, \nu)$*-separated with respect to* $f$ if all distinct points $x, y \in S$ are $(f, \delta, \nu)$-separated. The *(asymptotic) separation number* $\mathrm{Sep}(f, \delta, \nu)$, for distance $\delta > 0$ and frequency $\nu \in (0, 1)$, is then defined as the largest cardinality of a $(f, \delta, \nu)$-separated set in $X$. If these quantities are finite for all $\delta, \nu > 0$, we say $f$ has *finite separation numbers*, otherwise we say it has *infinite separation numbers*. Further, if there exists $\delta > 0$ such that $\mathrm{Sep}(f, \delta, \nu)$ is uniformly bounded in $\nu$ we say that $f$ has *bounded separation numbers*, otherwise we say *separation numbers are unbounded*. These notions provide a first qualitative indication concerning the complexity of a system. Roughly spoken, finite but unbounded separation numbers correspond to dynamics of intermediate complexity, which we are mainly interested in here. Once a system behaves 'chaotically', in the sense of positive entropy or weak mixing, separation numbers become infinite.

**Theorem 1.** *Suppose* $X$ *is a compact metric space and* $f : X \to X$ *is continuous. If* $f$ *has positive topological entropy or is weakly mixing with respect to some invariant probability measure* $\mu$*, then it has infinite separation numbers.*

Obviously, if $f$ is an isometry or, more generally, equicontinuous, then its separation numbers are bounded. Moving away from equicontinuity, one encounters the class of almost automorphic systems, which are central objects of study in topological dynamics and include many examples of both theoretical and practical importance [Aus88]. At least in the minimal case, separation numbers are suited to describe this transition, as the following result shows. In order to state it, suppose that $(X, d)$ and $(\Xi, \rho)$ are metric spaces and $f : X \to X$ and $g : \Xi \to \Xi$ are continuous. We say that $f$ is an extension of $g$ if there exists a continuous onto map $h : X \to \Xi$ such that $h \circ f = g \circ h$. The map $f$ is called an *almost 1-1 extension* of $g$ if there exists $y \in \Xi$ with $\#h^{-1}(y) = 1$. We further say it is an *almost sure 1-1 extension* if the set $E = \{y \in \Xi \mid \#h^{-1}(y) > 1\}$ has measure zero with respect to every $g$-invariant probability measure $\mu$ on $\Xi$.[1] Due to Veech's structure theorem [Vee65], almost automorphic minimal systems can be defined as almost 1-1 extensions of equicontinuous minimal systems.

**Theorem 2.** *Suppose $X$ is a compact metric space and $f : X \to X$ is a homeomorphism.*

*(a) If $f$ is minimal and almost automorphic, but not equicontinuous, then $f$ has unbounded separation numbers.*

*(b) If $f$ is an almost sure 1-1 extension of an equicontinuous system, then $f$ has finite separation numbers.*

Two examples for case (b) which we discuss in more detail are regular Toeplitz flows and Delone dynamical systems arising from certain cut and project quasicrystals.

In order to obtain quantitative information, we proceed to study the scaling behaviour of separation numbers as the separation frequency $\nu$ goes to zero. In principle, one may consider arbitrary growth rates. However, as the examples we discuss all indicate, it is polynomial growth which is the most relevant. Given $\delta > 0$, we let

$$\overline{\mathrm{ac}}(f,\delta) \;=\; \varlimsup_{\nu\to 0} \frac{\log \mathrm{Sep}(f,\delta,\nu)}{-\log \nu} \;, \qquad \underline{\mathrm{ac}}(f,\delta) \;=\; \varliminf_{\nu\to 0} \frac{\log \mathrm{Sep}(f,\delta,\nu)}{-\log \nu} \tag{3}$$

and define the *upper*, respectively *lower amorphic complexity of $f$* as

$$\overline{\mathrm{ac}}(f) \;=\; \sup_{\delta>0} \overline{\mathrm{ac}}(f,\delta) \quad \text{and} \quad \underline{\mathrm{ac}}(f) \;=\; \sup_{\delta>0} \underline{\mathrm{ac}}(f,\delta)\ . \tag{4}$$

If both values coincide, $\mathrm{ac}(f) = \overline{\mathrm{ac}}(f) = \underline{\mathrm{ac}}(f)$ is called the *amorphic complexity* of $f$. We have the following basic properties.

[1] Note that equicontinuous minimal systems are uniquely ergodic, such that there is only one measure to consider in this case.

**Proposition 3.** *Suppose $X, \Xi$ are compact metric spaces and $f : X \to X$, $g : \Xi \to \Xi$ continuous maps. Then the following statements hold.*

*(a)* Factor relation: *If $g$ is a factor of $f$, then $\overline{\mathrm{ac}}(f) \geq \overline{\mathrm{ac}}(g)$ and $\underline{\mathrm{ac}}(f) \geq \underline{\mathrm{ac}}(g)$. In particular, amorphic complexity is an invariant of topological conjugacy.*

*(b)* Power invariance: *For all $m \in \mathbb{N}$ we have $\overline{\mathrm{ac}}(f^m) = \overline{\mathrm{ac}}(f)$ and $\underline{\mathrm{ac}}(f^m) = \underline{\mathrm{ac}}(f)$.*

*(c)* Product formula: *If upper and lower amorphic complexity coincide for both $f$ and $g$, then the same holds for $f \times g$ and we have $\mathrm{ac}(f \times g) = \mathrm{ac}(f) + \mathrm{ac}(g)$. Otherwise, we have $\overline{\mathrm{ac}}(f \times g) \leq \overline{\mathrm{ac}}(f) + \overline{\mathrm{ac}}(g)$ and $\underline{\mathrm{ac}}(f \times g) \geq \underline{\mathrm{ac}}(f) + \underline{\mathrm{ac}}(g)$.*

*(d)* Commutation invariance: $\overline{\mathrm{ac}}(f \circ g) = \overline{\mathrm{ac}}(g \circ f)$ *and* $\underline{\mathrm{ac}}(f \circ g) = \underline{\mathrm{ac}}(g \circ f)$.

As the power invariance indicates, amorphic complexity behaves quite differently than topological entropy in some aspects. In this context, it should also be noted that no variational principle can be expected for amorphic complexity. This is a direct consequence of the requirement (iii) above, which is met by amorphic complexity (see below). The reason is that since Sturmian subshifts and Denjoy examples are uniquely ergodic and measure-theoretically isomorphic to a irrational rotations, they cannot be distinguished on a measure-theoretic level. Hence, no reasonable analogue to the variational principle for topological entropy can be satisfied.

**Proposition 4.** *Amorphic complexity is zero for all isometries and Morse-Smale-systems, but equals one for Sturmian subshifts and Denjoy examples on the circle.*

The arguments in the proof of Theorem 2 can be quantified, at least to some extent, to obtain an upper bound on amorphic complexity for minimal almost sure 1-1 extensions of isometries. In rough terms, the results reads as follows. By $\overline{\mathrm{Dim}}_B(A)$ and $\underline{\mathrm{Dim}}_B(A)$ we denote the upper and lower box dimension, respectively, of a subset $A$ of a metric space. If both quantities coincide we denote the common value by $\mathrm{Dim}_B(A)$ and say the box dimension of $A$ is well-defined.

**Theorem 5.** *Suppose $X$ and $\Xi$ are compact metric spaces and $f : X \to X$ is an almost sure 1-1 extension of a minimal isometry $g : \Xi \to \Xi$, with factor map $h$. Further, assume that the box dimension of $\Xi$ is well-defined. Then*

$$\overline{\mathrm{ac}}(f) \;\leq\; \frac{\gamma(h) \cdot \mathrm{Dim}_B(\Xi)}{\mathrm{Dim}_B(\Xi) - \sup_{\delta > 0} \mathrm{Dim}_B(E_\delta)} \;, \tag{5}$$

*where $E_\delta = \{\xi \in \Xi \mid \mathrm{diam}(h^{-1}(\xi) \geq \delta\}$ and $\gamma(h)$ is a scaling factor depending on the local properties of the factor map $h$.*

It should be mentioned, however, that at least according to our current understanding, this result is of rather abstract nature. The reason is the fact that the scaling factor $\gamma(h)$ seems to be difficult to determine in specific examples, where we use direct methods instead to obtain improved explicit estimates.

In this direction, we first investigate *regular Toeplitz flows*. Given a finite alphabet $A$, a sequence $\omega \in A^{\mathbb{T}}$ with $\mathbb{T} = \mathbb{N}$ or $\mathbb{Z}$ is called *Toeplitz*, if for all $n \in \mathbb{T}$ there exists $p \in \mathbb{N}$ such that $\omega_{n+kp} = \omega_n$ for all $k \in \mathbb{T}$. In other words, every symbol in a Toeplitz sequence occurs periodically. Thus, if we let $\mathrm{Per}(p,\omega) = \{n \in \mathbb{T} \mid \omega_{n+kp} = \omega_n \ \forall k \in \mathbb{T}\}$, then $\bigcup_{p\in\mathbb{N}} \mathrm{Per}(p,\omega) = \mathbb{T}$. By $D(p) = \frac{1}{p}\#(\mathrm{Per}(p,\omega) \cap [1,p])$ we denote the density of the $p$-periodic positions. If $\overline{\lim}_{p\to\infty} D(p) = 1$, then the Toplitz sequence is called *regular*. A sequence $(p_\ell)_{\ell\in\mathbb{N}}$ of integers such that $p_{\ell+1}$ is a multiple of $p_\ell$ for all $\ell \in \mathbb{N}$ and $\bigcup_{\ell\in\mathbb{N}} \mathrm{Per}(p_\ell,\omega) = \mathbb{T}$ is called a *weak periodic structure* for $\omega$. We denote the shift orbit closure of $\omega$ by $\Sigma_\omega$, such that $(\Sigma_\omega, \sigma)$ is the subshift generated by $\omega$. Then we have

**Theorem 6.** *Suppose $\omega$ is a regular Toeplitz sequence with weak periodic structure $(p_\ell)_{\ell\in\mathbb{N}}$. Then*

$$\mathrm{ac}(\sigma_{|\Sigma_\omega}) \ \leq \ \overline{\lim_{\ell\to\infty}} \frac{\log p_{\ell+1}}{\log(1 - D(p_\ell))} \ . \tag{6}$$

Moreover, it is possible to construct examples demonstrating that this estimate is sharp and that a dense set of values in $[1,\infty)$ is attained.

Finally, we turn to cut and project quasicrystals. Suppose $\tilde{L}$ is a cocompact discrete subgroup of $\mathbb{R}^n \times R^D$ such that $\pi_1 : \tilde{L} \to \mathbb{R}^n$ is injective and $\pi_2 : \tilde{L} \to \mathbb{R}^D$ has dense image. Further, assume that $W \subseteq \mathbb{R}^D$ is compact and satisfies $W = \overline{\mathrm{int}(W)}$. The pair $(\tilde{L}, W)$ is called a *cut and project scheme* and defines a Delone subset $\Lambda(W) = \pi_1\left((\mathbb{R}^m \times W) \cap \tilde{L}\right)$ of $\mathbb{R}^m$. A natural $\mathbb{R}^m$-action on the space of Delone sets in $\mathbb{R}^m$ is given by $(t,\Lambda) \mapsto \Lambda - t$. Taking the orbit closure $\Omega(\Lambda(W)) = \overline{\{\Lambda(W) - t \mid t \in \mathbb{R}^m\}}$ of $\Lambda(W)$, in a suitable topology, we obtain a *Delone dynamical system* $(\Omega(\Lambda(W)), \mathbb{R}^m)$ whose dynamical properties are closely related to the geometry of the Delone set $\Lambda(W)$. We refer to [Sch99, Moo00, LP03, BLM07] and references therein for further details. For the amorphic complexity, adapted to general actions of amenable groups, we obtain

**Theorem 7** ([FGJ])**.** *Suppose $(\tilde{L}, W)$ is a cut and project scheme in $\mathbb{R}^D \times \mathbb{R}^m$ and $(\Omega(W), \mathbb{R}^m)$ is the associated Delone dynamical system. Then*

$$\mathrm{ac}(\Omega, \mathbb{R}^m) \ \leq \ \frac{D}{D - \mathrm{Dim}_B(W)} \ . \tag{7}$$

As in the case of regular Toeplitz flows, it can be demonstrated by means of examples that this estimate is sharp. At the same time, equality does not always hold.

It is well-known that under the above assumptions the dynamical system $(\Omega(\Lambda(W)), \mathbb{R}^m)$ is a almost 1-1 extension of a minimal and isometric $\mathbb{R}^m$-action on a $D$-dimensional torus. Moreover, it turns out that with the notions of Theorem 2 we have $\mathrm{Dim}_B(W) = \mathrm{Dim}_B(E_\delta)$ for all $\delta > 0$. Thus, (7) can be interpreted as a special case of (5), with $\gamma(h) = 1$. However, as we have mentioned, the proof is independent and based on more direct arguments. Hence, while the presented results point clearly show some close relations, a better understanding of the underlying structures still has to be obtained and should be the aim of future research.

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