LYAPUNOV EXPONENTS OF NON-ARITHMETIC COMPLEX HYPERBOLIC LATTICES

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(joint work with Martin Möller) To a flat vector bundle $\mathbb{V}$ over a Riemannian manifold B, one can associate its Lyapunov exponents

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m,$$

the different mean logarithmic growth rates of sections when parallel transported along the geodesic flow. In complex geometry, naturally occuring flat vector bundles are the relative cohomology bundles $\mathbb{V} = R^1 f_* \mathbb{C}$ of a family $f : \mathcal{X} \to B$ of curves (or more generally of a family of Kähler manifolds). In this case, the flat bundles in question are naturally endowed with a relative Hodge filtration, i.e. they are variations of Hodge structures, which are moreover polarized by the cup product pairing on cohomology.

In the case of a family of curves over a hyperbolic curve, there is a beautiful formula, first discovered by Kontsevich [Kon97] (see also [EKZ10]), that relates the sum of Lyapunov exponents to the degrees of certain line bundles. We show a variant of this formula, where the base is a ball quotient, the orbit space of a lattice Γ acting on complex hyperbolic n-space $\mathbb{B}^n$.

Theorem 1 ([KM12]). *Let $\mathbb{V}$ be a real polarized variation of Hodge structures of weight 1 and rank $2k$ on a ball quotient $B = \mathbb{B}^n/\Gamma$, and let $\mathcal{V}^{1,0}$ be its $(1,0)$-subbundle. Then the $2k$ Lyapunov exponents of $\mathbb{V}$ (repeated according to their multiplicities) satisfy*

$$\lambda_1 + \cdots + \lambda_k = \frac{(n+1)c_1(\mathcal{V}^{1,0}).c_1(\omega_B)^{n-1}}{c_1(\omega_B)^n},$$

where ω_B denotes the canonical bundle.

The most prominent examples of non-arithmetic complex hyperbolic lattices were found by Picard, Terada, Deligne, Mostow and Thurston. Their ball quotients parametrize cyclic coverings of the line and thus come naturally with a flat vector bundle carrying a variation of Hodge structures. Using the above formula combined with the symmetry of the Lyapunov spectrum and other considerations, we can effectively compute all individual Lyapunov exponents of all Picard-Terada-Deligne-Mostow-Thurston examples.

As a second result, we show that Lyapunov exponents provide commensurability invariants of complex hyperbolic lattices. Together with the above computations and using previous considerations (see [Par09]), we conclude that the non-arithmetic Picard-Terada-Deligne-Mostow-Thurston examples fall into 10 commensurability classes.

References

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