Subdiagrams and invariant measures on Bratteli diagrams

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1 Introduction and background

We study finite and infinite ergodic measures on Bratteli diagrams invariant with respect to the tail equivalence relation. Our aim is to characterize those subdiagrams that support an ergodic finite invariant measure.

Bratteli diagram is an important object in the theories of operator algebras and dynamical systems. It was originally defined in 1972 by O. Bratteli [4] for classification of AF C^*-algebras. Bratteli diagrams turned out to be a powerful tool for the study of measurable, Borel, and Cantor dynamics (see [8, 6, 5, 7]).

A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i \geq 0} V_i$ and the edge set $E = \bigcup_{i \geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i where

(i) $V_0 = \{v_0\}$ is a single point;

(ii) V_i and E_i are finite sets;

(iii) there exist a range map r and a source map s, both from E to V, such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

The n-th *incidence matrix* of a Bratteli diagram B is a $|V_{n+1}| \times |V_n|$ matrix $F_n = (f^{(n)}_{v,w})$, $n \geq 0$, such that $f^{(n)}_{v,w} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|$ for $v \in V_{n+1}$ and $w \in V_n$. Here the symbol $|\cdot|$ denotes the cardinality of a set.

Denote by X_B the set of all infinite paths in $B = (V, E)$, which start from v_0. The topology defined by finite paths (cylinder sets) turns X_B into a 0-dimensional metric compact space. We consider only such Bratteli diagrams for which X_B is a *Cantor set*.

Two paths $x = (x_n)$ and $y = (y_n)$ in X_B are *tail (cofinal) equivalent* if and only if $x_n = y_n$ for n sufficiently large (notation: $x\mathcal{E}y$).

We use also the following notation for an $\mathcal{E}$-invariant measure μ on X_B and $n \geq 1$ and $v \in V_n$:

- $X^{(n)}_v \subset X_B$ denotes the set of all paths that go through the vertex v;
- $h^{(n)}_v$ denotes the cardinality of the set of all finite paths (cylinder sets) between v_0 and v;

- $p_v^{(n)}$ denotes the μ-measure of the cylinder set $e(v_0, v)$ corresponding to a finite path between v_0 and v (since μ is $\mathcal{E}$-invariant, the value $p_v^{(n)}$ does not depend on $e(v_0, v)$).

Define the n-th stochastic $|V_{n+1}| \times |V_n|$ matrix Q_n:

$$q_{v,w}^{(n)} = f_{v,w}^{(n)} \frac{h_w^{(n)}}{h_v^{(n+1)}}, \quad v \in V_{n+1}, \ w \in V_n.$$

A (vertex) subdiagram $\overline{B} = (\overline{W}, \overline{E})$ of B is a Bratteli diagram formed by the vertices from $W_n \subset V_n$ and by the set of edges $\overline{E}$ such that the incidence matrix $\overline{F}_n$ is determined by those edges from B that have their source and range in vertices from W_n and W_{n+1}, respectively. Denote $W'_n = V_n \setminus W_n$. As a rule, objects related to a subdiagram $\overline{B}$ are denoted by barred symbols.

Let $\widehat{X}_{\overline{B}} = \mathcal{E}(X_{\overline{B}})$ be the subset of paths in X_B that are tail equivalent to paths from $X_{\overline{B}}$. Let $\overline{\mu}$ be a probability measure on $X_{\overline{B}}$ invariant with respect to the tail equivalence relation defined on $\overline{B}$. Then $\overline{\mu}$ can be canonically extended to the measure $\widehat{\overline{\mu}}$ on the space $\widehat{X}_{\overline{B}}$ by invariance with respect to $\mathcal{E}$. To extend $\widehat{\overline{\mu}}$ to the whole space X_B, set $\widehat{\overline{\mu}}(X_B \setminus \widehat{X}_{\overline{B}}) = 0$.

A Bratteli diagram $B = (V, E)$ is called of *finite rank* if there exists $d \in \mathbb{N}$ such that $|V_n| \le d$ for every n. In particular, if all incidence matrices of B are the same, then B is called *stationary*. Ergodic invariant measures on stationary Bratteli diagrams can be explicitly described by the incidence matrix of the diagram [1].

The motivation for this work arises from the following result in [3]:

Theorem 1.1 (Bezuglyi-Kwiatkowski-Medynets-Solomyak, 2013)**.** *For any ergodic probability measure μ on a finite rank diagram B, there exists a subdiagram $\overline{B}$ of B defined by a sequence $W = (W_n)$, where $W_n \subset V_n$, such that $\mu(X_w^{(n)})$ is bounded from zero for all $w \in W_n$ and n. It was also shown that μ can be obtained as an extension of an ergodic measure on the subdiagram $\overline{B}$, in other words, $\overline{B}$ supports μ.*

2 Finiteness of measure extension

Given a Bratteli diagram B and a subdiagram $\overline{B}$ of B, we find necessary and sufficient conditions for finiteness of measure extension $\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}})$ of a probability measure $\overline{\mu}$ defined on $\overline{B}$. The following Proposition 2.1 and Theorem 2.2 can be found in [2].

Proposition 2.1. *Let B be a Bratteli diagram with incidence stochastic matrices $\{Q_n = (q_{v,w}^{(n)})\}$ and let $\overline{B}$ be a proper vertex subdiagram of B defined by a sequence of subsets (W_n) where $W_n \subset V_n$. If*

$$\sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \sum_{v \in W'_n} q_{w,v}^{(n)} < \infty,$$

then any probability measure $\overline{\mu}$ defined on the path space $X_{\overline{B}}$ of the subdiagram $\overline{B}$ extends to a finite measure $\widehat{\overline{\mu}}$ on $\widehat{X}_{\overline{B}}$.

Theorem 2.2. *Let B be a Bratteli diagram with the sequence of incidence matrices $\{F_n\}_{n=0}^{\infty}$ and $\overline{B}$ be a vertex subdiagram of B defined by the sequence of subsets (W_n), $W_n \subset V_n$. Suppose $\overline{\mu}$ is a probability invariant measure on $X_{\overline{B}}$. Then the extension $\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}})$ is finite if and only if*

$$\sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \widehat{\overline{\mu}} \left(X_w^{(n+1)} \right) \sum_{v \in W'_n} q_{w,v}^{(n)} < \infty,$$

or, equivalently,

$$\sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \sum_{v \in W'_n} f_{wv}^{(n)} h_v^{(n)} \overline{p}_w^{(n+1)} < \infty,$$

where $\overline{p}_v^{(n)}$ denotes the $\overline{\mu}$-measure of the cylinder set corresponding to a finite path between v_0 and $v \in W_n$.

We now consider in detail the class of Bratteli diagrams of rank two. It turns out that this class can be studied completely, and one can find necessary and sufficient conditions for a diagram to have a single finite ergodic measure or two finite ergodic measures.

Proposition 2.3. *Let B be a Bratteli diagram with 2×2 incidence matrices F_n such that*

$$F_n = \begin{pmatrix} a_n & c_n \\ d_n & b_n \end{pmatrix},$$

where $a_n + c_n = d_n + b_n = r_n$ for every n. Then there are two finite ergodic invariant measures if and only if

$$\sum_{k=1}^{\infty} \left(1 - \frac{|a_k - d_k|}{r_k} \right) < \infty.$$

There is a unique invariant measure μ on B if and only if

$$\sum_{k=1}^{\infty} \left(1 - \frac{|a_k - d_k|}{r_k} \right) = \infty.$$

In particular, if one of the series $\sum_{k=1}^{\infty} \left(1 - \frac{\max\{a_k, d_k\}}{r_k} \right)$ and $\sum_{k=1}^{\infty} \frac{\min\{a_k, d_k\}}{r_k}$ converges and the other diverges, then the unique invariant measure μ is an extension of a measure from an odometer (a Bratteli diagram with one vertex on each level). Otherwise, if both of the series diverge, then there is no odometer such that μ is an extension of a measure from this odometer.

As known, any Bratteli diagram of rank k can have at most k ergodic measures [3]. The next result gives conditions under which such a diagram has exactly k ergodic measures.

Theorem 2.4. *Let $B = (V, E)$ be a Bratteli diagram of rank k. Let $F_n = (f_{v,w}^{(n)})$ be incidence matrices for B such that $\sum_{w \in V_n} f_{v,w}^{(n)} = r_n \geq 2$ for every $v \in V_{n+1}$ and* $\det F_n \neq 0$ *for every n. Let*

$$z^{(n)} = \det \begin{pmatrix} \frac{f_{1,1}^{(n)}}{r_n} & \dots & \frac{f_{1,k-1}^{(n)}}{r_n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{f_{k,1}^{(n)}}{r_n} & \dots & \frac{f_{k,k-1}^{(n)}}{r_n} & 1 \end{pmatrix}.$$

Then there exist exactly k ergodic invariant measures on B if and only if the product $\prod_{n=1}^{\infty} |z^{(n)}|$ converges.

3 Measure of the path space of a subdiagram

Let $\overline{B}$ be a subdiagram of a Bratteli diagram B. Suppose that a probability measure μ is given on B. The question is when the path space $X_{\overline{B}}$ of the subdiagram $\overline{B}$ considered as a subset of X_B has positive measure μ.

Denote by $Y_w^{(n)}$ the set of all paths $x = (x_1, \dots, x_n, \dots)$ from X_B which pass through vertex $w \in W_n$ and such that the finite path $(x_1, \dots, x_n)$ lies in $\overline{B}$. Set $Y^{(n)} = \bigcup_{w \in W_n} Y_w^{(n)}$.

Theorem 3.1. *Let $\overline{B}$ be a vertex subdiagram of B, and μ, $Y^{(n)}$ be as above. Then the series*

$$S = \sum_{n=1}^{\infty} \sum_{v \in W'_{n+1}} \sum_{w \in W_n} f_{vw}^{(n)} p_v^{(n+1)} \overline{h}_w^{(n)}$$

is always convergent and $\mu(X_{\overline{B}}) = \mu(Y^{(1)}) - S$. Hence, $\mu(X_{\overline{B}}) = 0 \iff S = \mu(Y^{(1)})$.

The following theorem gives a necessary and sufficient condition for a subdiagram $\overline{B}$ of B to have a path space of zero measure in X_B. This is one of our main results.

Theorem 3.2. *Let B be a simple Bratteli diagram and μ a probability ergodic measure on X_B. Suppose $\overline{B}$ is a vertex subdiagram of B defined by a sequence (W_n) of vertices subsets. Then $\mu(X_{\overline{B}}) = 0$ if and only if for every $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that for every $w \in W_n$*

$$\frac{\overline{h}_w^{(n)}}{h_w^{(n)}} < \varepsilon.$$

Corollary 3.3. *Let B, μ, $\overline{B}$ be as in Theorem 3.2. Suppose $\mu(X_{\overline{B}}) = 0$. Then for any probability invariant $\overline{\mu}$ on $\overline{B}$ we have $\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) = \infty$.*

Proof. Assume that the converse holds, i.e., there exists M such that $\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) < M$. Take $\varepsilon > 0$ such that $\frac{1}{\varepsilon} > M$. Given $\varepsilon > 0$, we can find $n = n(\varepsilon)$, by Theorem 3.2, such that $\dfrac{\overline{h}_w^{(n)}}{h_w^{(n)}} < \varepsilon$ for every $w \in W_n$. Then

$$
\begin{aligned}
\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) &> \sum_{w \in W_n} h_w^{(n)} \overline{p}_w^{(n)} \\
&= \sum_{w \in W_n} \frac{h_w^{(n)}}{\overline{h}_w^{(n)}} \overline{h}_w^{(n)} \overline{p}_w^{(n)} \\
&> \frac{1}{\varepsilon} \sum_{w \in W_n} \overline{h}_w^{(n)} \overline{p}_w^{(n)} \\
&> M.
\end{aligned}
$$

This is a contradiction. $\square$

4 Example

Let B be a Bratteli diagram defined by rectangular incidence matrices

$$F_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

Then there is a unique probability invariant measure μ on X_B.

Let $\overline{B}$ be a vertex subdiagram of B defined by a sequence of vertices (W_n), where $W_n \subsetneq V_n$. Then

$$\mu(X_{\overline{B}}) > 0 \iff \prod_{i=1}^{\infty} \frac{|W_i|}{|V_i|} > 0 \iff \sum_{i=1}^{\infty} \left(1 - \frac{|W_i|}{|V_i|}\right) < \infty.$$

Let $\overline{\mu}$ be the unique invariant measure on $X_{\overline{B}}$. Then

$$\left(\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) < \infty\right) \iff \left(\prod_{i=1}^{\infty} \frac{|V_i|}{|W_i|} < \infty\right).$$

Hence

$$\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) < \infty \iff \mu(X_{\overline{B}}) > 0.$$

In particular, if $\overline{B}$ or B are of finite rank, then $\mu(X_{\overline{B}}) = 0$.

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