# Minimal Data Rates and Entropy for Control Problems

Christoph Kawan

Courant Institute of Mathematical Sciences
New York University

Max-Planck-Institut Bonn
Activity “Dynamics and Numbers”
July 16, 2014

# Outline

# Digitally networked control systems

Examples for networked systems:

- automated highway systems (vehicle platoons)
- sensor networks (e.g., smart cities)
- unmanned aerial vehicles
- smart grids
- telerobotics
- ...

# Communication in digitally networked systems

It is not always possible to transmit information instantaneously, lossless and with arbitrary precision. This raises the question about the minimal information per time unit (data rate) necessary to accomplish a certain control task.

# Communication in digitally networked systems

It is not always possible to transmit information instantaneously, lossless and with arbitrary precision. This raises the question about the minimal information per time unit (data rate) necessary to accomplish a certain control task.

To analyze this problem, start with the simplest setting:

# Communication in digitally networked systems

It is not always possible to transmit information instantaneously, lossless and with arbitrary precision. This raises the question about the minimal information per time unit (data rate) necessary to accomplish a certain control task.

To analyze this problem, start with the simplest setting:

**The simplest network topology**

One controller and one dynamical system connected via a digital channel with a certain bit rate

# Communication in digitally networked systems

It is not always possible to transmit information instantaneously, lossless and with arbitrary precision. This raises the question about the minimal information per time unit (data rate) necessary to accomplish a certain control task.

To analyze this problem, start with the simplest setting:

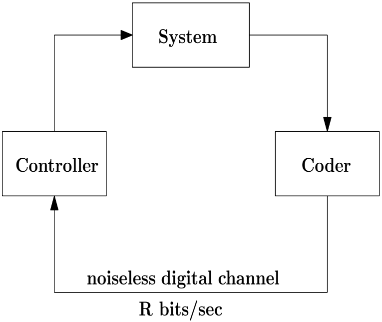
**The simplest network topology**

One controller and one dynamical system connected via a digital channel with a certain bit rate

**The simplest control problem**

Invariance of a set (example: vehicle platoons)

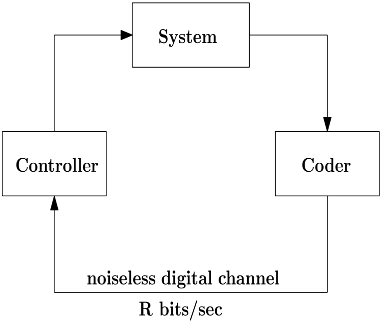
# The simplest setting




## Explanation

**System** Deterministic, discrete or continuous time
**Coder** Encodes the state by a symbol from a (time-dependent) alphabet at discrete times $k\tau$, $k = 0, 1, 2, \dots$
**Controller** Generates open-loop controls on a finite time interval of length $\tau$

# The simplest setting




## Explanation

**System** Deterministic, discrete or continuous time
**Coder** Encodes the state by a symbol from a (time-dependent) alphabet at discrete times $k\tau$, $k = 0, 1, 2, \ldots$
**Controller** Generates open-loop controls on a finite time interval of length $\tau$

## Control objective

Invariance of a compact subset of the state space

# Feedback entropy (Nair, Evans, Mareels, Moran 2004)

Consider a discrete-time system with transition map

$$\varphi : \mathbb{Z}_+ \times X \times U^{\mathbb{Z}_+} \to X, \qquad (n, x, \omega) \mapsto \varphi(n, x, \omega).$$

Assumption: $Q \subset X$ a compact and controlled invariant set, i.e.,

$$\forall x \in Q \; \exists \omega \in U^{\mathbb{Z}_+} : \; \forall k \geq 1, \; \varphi(k, x, \omega) \in \operatorname{int} Q.$$

# Feedback entropy (Nair, Evans, Mareels, Moran 2004)

Consider a discrete-time system with transition map

$$\varphi : \mathbb{Z}_+ \times X \times U^{\mathbb{Z}_+} \to X, \qquad (n, x, \omega) \mapsto \varphi(n, x, \omega).$$

Assumption: $Q \subset X$ a compact and controlled invariant set, i.e.,

$$\forall x \in Q \; \exists \omega \in U^{\mathbb{Z}_+} : \; \forall k \geq 1, \; \varphi(k, x, \omega) \in \operatorname{int} Q.$$

## Invariant open covers

A triple $(\mathcal{A}, \tau, G)$, where $\mathcal{A}$ an open cover of $Q$, $\tau \geq 1$, and $G : \mathcal{A} \to U^\tau$ s.t.

$$\varphi(k, A, G(A)) \subset \operatorname{int} Q, \quad A \in \mathcal{A}, \quad k = 1, \ldots, \tau.$$

Every finite sequence ("path") of sets in $\mathcal{A}$ defines an open set consisting of the points which follow this path under the dynamics given by $(\mathcal{A}, \tau, G)$. This yields a sequence $\mathcal{A}^{[n]}$ of open covers of $Q$.

# Feedback entropy

The entropy of $(\mathcal{A}, \tau, G)$ is given by

$$h_{\mathrm{fb}}(\mathcal{A}, \tau, G) := \lim_{n \to \infty} \frac{1}{n\tau} \log_2 N(Q|\mathcal{A}^{[n]}),$$

where $N(Q|\mathcal{A}^{[n]})$ is the minimal cardinality of a subcover. The feedback entropy of $Q$ is defined as

$$h_{\mathrm{fb}}(Q) := \inf_{(\mathcal{A}, \tau, G)} h_{\mathrm{fb}}(\mathcal{A}, \tau, G).$$

# Alternative definition: Invariance entropy (Colonius 2007)

A set $\mathcal{S}$ of control sequences is called $(\tau, Q)$-spanning if

$$\forall x \in Q \; \exists \omega \in \mathcal{S} : \;\; \varphi(k, x, \omega) \in \operatorname{int} Q, \;\; k = 1, \ldots, \tau.$$

If $r_{\mathrm{inv}}(\tau, Q)$ is the minimal cardinality of such a set, then

$$h_{\mathrm{fb}}(Q) = \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\mathrm{inv}}(\tau, Q).$$

# Alternative definition: Invariance entropy (Colonius 2007)

A set $\mathcal{S}$ of control sequences is called $(\tau, Q)$-spanning if

$$\forall x \in Q \;\exists \omega \in \mathcal{S}: \;\; \varphi(k, x, \omega) \in \operatorname{int} Q, \;\; k = 1, \ldots, \tau.$$

If $r_{\mathrm{inv}}(\tau, Q)$ is the minimal cardinality of such a set, then

$$h_{\mathrm{fb}}(Q) = \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\mathrm{inv}}(\tau, Q).$$

Advantage of this characterization: Simpler and easier to adapt to other control problems!

# Alternative definition: Invariance entropy (Colonius 2007)

A set $\mathcal{S}$ of control sequences is called $(\tau, Q)$-spanning if

$$\forall x \in Q \; \exists \omega \in \mathcal{S} : \; \varphi(k, x, \omega) \in \operatorname{int} Q, \; k = 1, \ldots, \tau.$$

If $r_{\mathrm{inv}}(\tau, Q)$ is the minimal cardinality of such a set, then

$$h_{\mathrm{fb}}(Q) = \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\mathrm{inv}}(\tau, Q).$$

Advantage of this characterization: Simpler and easier to adapt to other control problems!

**Intuition**

If the controller receives $n$ bits per unit time, it can generate at most $2^n$ different control sequences. Therefore, the number of control sequences, necessary to accomplish the control task for arbitrary initial states on a finite time interval, is a measure for the necessary information.

# Additional assumptions

**Goal**

Describe $h_{\mathrm{fb}}(Q)$ in terms of dynamical quantities such as Lyapunov exponents

# Additional assumptions

**Goal**

Describe $h_{\rm fb}(Q)$ in terms of dynamical quantities such as Lyapunov exponents

To this end, we need more control-theoretic and dynamical structure!

- Control-theoretic: Global and infinitesimal controllability properties
- Dynamical: Hyperbolicity

# Additional assumptions

## Goal

Describe $h_{\mathrm{fb}}(Q)$ in terms of dynamical quantities such as Lyapunov exponents

To this end, we need more control-theoretic and dynamical structure!

- Control-theoretic: Global and infinitesimal controllability properties
- Dynamical: Hyperbolicity

## From now on:

Continuous and time-invertible systems, $\varphi : \mathbb{R} \times M \times \mathcal{U} \to M$, where $M$ a differentiable manifold and $\varphi(\cdot, x, \omega)$ solution of a differential equation

$$\dot{x}(t) = F(x(t), \omega(t)), \quad \omega \in \mathcal{U} = L^{\infty}(\mathbb{R}, U), \ U \subset \mathbb{R}^m.$$

# Controllability properties

**Control set (Colonius, Kliemann)**

A set $D \subset M$ with nonempty interior is called a control set if it is a maximal set of complete approximate controllability.

# Controllability properties

## Control set (Colonius, Kliemann)

A set $D \subset M$ with nonempty interior is called a control set if it is a maximal set of complete approximate controllability.

## Regular trajectories

A trajectory $(\varphi(\cdot, x, \omega), \omega(\cdot))$ is called regular on $[0, \tau]$ if the linearization of the system along this trajectory is controllable on $[0, \tau]$.

Note: Regularity implies local controllability.

# Upper bounds

### Theorem [K., 2009]

Let $D$ be a control set with compact closure, and let $(\varphi(\cdot,x,\omega),\omega(\cdot))$ be a regular periodic trajectory in the interior of $D$. Then

$$h_{\mathrm{fb}}(\mathrm{cl}D) \leq \sum_{\lambda} \max\{0, n_\lambda \lambda\},$$

where the sum runs over the distinct Lyapunov exponents $\lambda$ with multiplicities $n_\lambda$.

# Upper bounds

## Theorem [K., 2009]

Let $D$ be a control set with compact closure, and let $(\varphi(\cdot, x, \omega), \omega(\cdot))$ be a regular periodic trajectory in the interior of $D$. Then

$$h_{\mathrm{fb}}(\mathrm{cl} D) \leq \sum_{\lambda} \max\{0, n_\lambda \lambda\},$$

where the sum runs over the distinct Lyapunov exponents $\lambda$ with multiplicities $n_\lambda$.

## Idea of proof

First steer into a small neighborhood of $x$ and then use local controllability to stay in a neighborhood of the periodic orbit for all future times. The sum of the positive Lyapunov exponents measures how fast one is driven away (on average) from the periodic trajectory without applying controls.

# Upper bounds

**Question**

When do regular periodic trajectories exist?

# Upper bounds

## Question

When do regular periodic trajectories exist?

## Answer

From a theorem of J.-M. Coron follows: If the system is smooth and satisfies a strong accessibility assumption ("strong jet accessibility"), then at every point in the interior of $D$ we find a regular periodic trajectory. For many classes of systems weaker conditions suffice.

# Upper bounds

## Corollary [K., 2013]

If the assumption for existence of regular trajectories is satisfied, then

$$h_{\mathrm{fb}}(\mathrm{cl}D) \leq \inf_{(\omega,x)} \limsup_{\tau\to\infty} \frac{1}{\tau} \log^{+} \|(\mathrm{d}\varphi_{\tau,\omega})_x^{\wedge}\|,$$

where the infimum runs over all $(\omega, x) \in \mathcal{U} \times M$ s.t. the corresponding trajectory $\varphi(\cdot, x, \omega)$ does not leave a compact subset of the interior of $D$.

# Lower bounds

## Definition

A compact and (full-time) controlled invariant set $Q$ is called hyperbolic if there is an invariant and continuous splitting

$$T_x M = E^s_{\omega,x} \oplus E^u_{\omega,x}$$

for all $(\omega, x)$ with $\varphi(\mathbb{R}, x, \omega) \subset Q$ such that uniform contraction on $E^s$ and uniform expansion on $E^u$ holds.

# Lower bounds

## Definition

A compact and (full-time) controlled invariant set $Q$ is called hyperbolic if there is an invariant and continuous splitting

$$T_xM = E^s_{\omega,x} \oplus E^u_{\omega,x}$$

for all $(\omega, x)$ with $\varphi(\mathbb{R}, x, \omega) \subset Q$ such that uniform contraction on $E^s$ and uniform expansion on $E^u$ holds.

## Remark

This makes sense for control-affine systems

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \omega_i(t) f_i(x(t))$$

with compact and convex control value space $U \subset \mathbb{R}^m$. In this setting, there exists a reasonable topology on $\mathcal{U}$.

# Lower bounds

## Estimation by escape rate

Let $\mu$ be a Borel measure on $M$ with $0 < \mu(Q) < \infty$. Then

$$h_{\mathrm{fb}}(Q) \geq \limsup_{\tau \to \infty} -\frac{1}{\tau} \log \sup_{\omega} \mu(Q(\omega,\tau)),$$

where

$$Q(\omega,\tau) = \{x \in Q : \ \varphi([0,\tau],x,\omega) \subset Q\}.$$

# Lower bounds

## Estimation by escape rate

Let $\mu$ be a Borel measure on $M$ with $0 < \mu(Q) < \infty$. Then

$$h_{\mathrm{fb}}(Q) \geq \limsup_{\tau\to\infty} -\frac{1}{\tau}\log\sup_{\omega}\mu(Q(\omega,\tau)),$$

where

$$Q(\omega,\tau) = \{x \in Q : \ \varphi([0,\tau],x,\omega) \subset Q\}.$$

From now on: $\mu = \mathrm{vol}$ (Riemannian volume). The measures of the sets $Q(\omega,\tau)$ are estimated by covering them with Bowen-balls

$$B_{\varepsilon}^{\omega,\tau}(x) = \{y \in M : \ d(\varphi(t,x,\omega),\varphi(t,y,\omega)) < \varepsilon, \ \forall t \in [0,\tau]\}.$$

# Volume lemma and lower estimate

## Bowen-Ruelle-Liu volume lemma

In the hyperbolic case, for small $\varepsilon$, all $\tau \geq 0$ and $(\omega, x)$ we have

$$0 < \alpha_\varepsilon \leq \mathrm{vol}(B_\varepsilon^{\omega,\tau}(x)) \left|\det(\mathrm{d}\varphi_{\tau,\omega})|_{E^u_{\omega,x}}\right| \leq \beta_\varepsilon < +\infty.$$

# Volume lemma and lower estimate

## Bowen-Ruelle-Liu volume lemma

In the hyperbolic case, for small $\varepsilon$, all $\tau \geq 0$ and $(\omega, x)$ we have

$$0 < \alpha_\varepsilon \leq \mathrm{vol}(B_\varepsilon^{\omega,\tau}(x)) \left|\det(\mathrm{d}\varphi_{\tau,\omega})|_{E^u_{\omega,x}}\right| \leq \beta_\varepsilon < +\infty.$$

## Theorem [Da Silva, K., 2014]

Let $Q$ be a hyperbolic set such that for each $\omega \in \mathcal{U}$ there exists a unique $x(\omega) \in Q$ with $\varphi(\mathbb{R}, x(\omega), \omega) \subset Q$. Then

$$h_{\mathrm{fb}}(Q) \geq \inf_{(\omega,x)\in\mathcal{Q}} \limsup_{\tau\to\infty} \frac{1}{\tau} \log \left|\det(\mathrm{d}\varphi_{\tau,\omega})|_{E^u_{\omega,x}}\right|,$$

where $\mathcal{Q} = \{(\omega, x) \in \mathcal{U} \times M : \varphi(\mathbb{R}, x, \omega) \subset Q\}$.

# Volume lemma and lower estimate

## Bowen-Ruelle-Liu volume lemma

In the hyperbolic case, for small $\varepsilon$, all $\tau \geq 0$ and $(\omega, x)$ we have

$$0 < \alpha_\varepsilon \leq \mathrm{vol}(B_\varepsilon^{\omega,\tau}(x)) \left| \det(\mathrm{d}\varphi_{\tau,\omega})|_{E^u_{\omega,x}} \right| \leq \beta_\varepsilon < +\infty.$$

## Theorem [Da Silva, K., 2014]

Let $Q$ be a hyperbolic set such that for each $\omega \in \mathcal{U}$ there exists a unique $x(\omega) \in Q$ with $\varphi(\mathbb{R}, x(\omega), \omega) \subset Q$. Then

$$h_{\mathrm{fb}}(Q) \geq \inf_{(\omega,x)\in\mathcal{Q}} \limsup_{\tau\to\infty} \frac{1}{\tau} \log \left| \det(\mathrm{d}\varphi_{\tau,\omega})|_{E^u_{\omega,x}} \right|,$$

where $\mathcal{Q} = \{(\omega, x) \in \mathcal{U} \times M : \varphi(\mathbb{R}, x, \omega) \subset Q\}$.

## Remark

The theorem applies to small control sets around hyperbolic equilibria.

# Right-invariant systems on flag manifolds

Let $G$ be a non-compact semisimple Lie group and $\mathbb{F}_\Theta = G/P_\Theta$ a flag manifold of $G$. Then a right-invariant control-affine system on $G$ induces a control-affine system on $\mathbb{F}_\Theta$. The control structure of such systems has been studied by Luiz San Martin and coworkers. In particular, there are finitely many control sets which are parametrized by a double coset space of the Weyl group of $G$.

# Right-invariant systems on flag manifolds

Let $G$ be a non-compact semisimple Lie group and $\mathbb{F}_\Theta = G/P_\Theta$ a flag manifold of $G$. Then a right-invariant control-affine system on $G$ induces a control-affine system on $\mathbb{F}_\Theta$. The control structure of such systems has been studied by Luiz San Martin and coworkers. In particular, there are finitely many control sets which are parametrized by a double coset space of the Weyl group of $G$.

**Theorem (Da Silva, K., 2014)**

Each control set $E_{\Theta,w}$ on $\mathbb{F}_\Theta$ has a partially hyperbolic structure. The hyperbolic ones satisfy

$$h_{\mathrm{fb}}(E_{\Theta,w}) = \inf_{(\omega,x)\in\mathcal{E}_{\Theta,w}} \limsup_{\tau\to\infty} \frac{1}{\tau} \log \left|\det(\mathrm{d}\varphi_{\tau,\omega})|_{E^u_{\omega,x}}\right|,$$

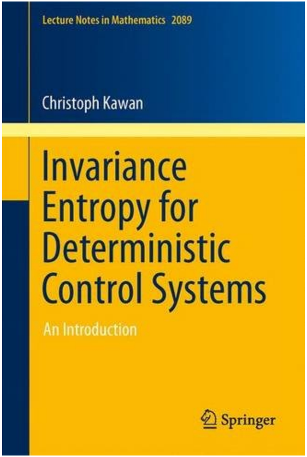
where $\mathcal{E}_{\Theta,w} = \{(\omega,x) : \varphi(\mathbb{R},x,\omega) \subset E_{\Theta,w}\}$.

# Right-invariant systems on flag manifolds

# Right-invariant systems on flag manifolds

**Remark**

For the non-hyperbolic control sets we need a better understanding of escape rates from small neighborhoods of invariant sets for non-autonomous dynamical systems. Even for autonomous systems there is still very little known for the non-hyperbolic cases.




**Thank you for your attention!**