STURMIAN COLORING OR REGULAR TREES

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Let T be a k-regular tree and G be the group of all automorphisms of T, which is a locally compact topological group with compact-open topology. By a colouring of a tree T, we mean a vertex coloring $\phi : VT \to \mathcal{A}$, where VT is a vertex set of T and $\mathcal{A}$ is a finite set. We define an invariant of a coloring ϕ called subword complexity.

A coloring $\phi : VT \to \mathcal{A}$ is *periodic* if there exists a subgroup $\Gamma \subset G$ such that $\Gamma \backslash T$ is a finite graph and ϕ is Γ-invariant, i.e., $\phi(\gamma x) = \phi(x)$, for all $x \in VT$ and $\gamma \in \Gamma$. Let Γ be a group acting on a k-regular tree T by automorphisms. If Γ acts without torsion, then the quotient $\Gamma \backslash T$ is a k-regular graph, but in general, the quotient has a structure of a graph of groups, a graph version of orbifold quotient.

For an infinite sequence u, the subword complexity $p_u(n)$ is defined as the number of different subwords of length n in u. Hedlund and Morse[2] showed that $p_u(n)$ is bounded if and only if u is eventually periodic. A sequence u is called Sturmian if $p_u(n) = n + 1$.

We define subword complexity $b_\phi(n)$ of a coloring ϕ as the number of colored n-balls in the tree colored by ϕ. We show that ϕ is periodic if and only if its subword complexity $b_\phi(n)$ is bounded. Then, we have an analogous theorem as follows

Theorem 1. *Let $\phi : VT \to \mathcal{A}$ be a coloring. The following are equivalent.*

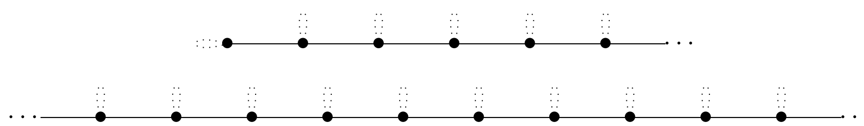
(1) *The coloring ϕ is periodic.*
(2) *The subword complexity of ϕ satisfies $b_\phi(n+1) = b_\phi(n)$ for some $n > 0$.*
(3) *The subword complexity $b_\phi(n)$ is bounded.*

For an example, let $\Gamma = \langle a_1, \cdots, a_k : a_i^2 = 1 \rangle$ and T be its Cayley graph. Any element g of G is associated a coloring ϕ_g as a permutation of colouring of neighboring vertices. The coloring ϕ_g is periodic if and only if g is an element of the commensurator of Γ[1, 3]. And as a corollary, an automorphism g of T is contained in the commensurator subgroup of Γ if and only if its subword complexity $b_{\phi_g}(n)$ is bounded.

We define Sturmian colorings as colorings with minimal unbounded subword complexity, i.e. with $b_\phi(n) = n + 2$, and study them using the type sets of vertices. The main result of this article is that any Sturmian coloring is a lifting of a coloring of a graph X, which is an infinite geodesic or a geodesic ray with loops possibly attached. With an additional condition of bounded type, it is a lifting of a coloring of a geodesic ray with loops possibly attached. We further give a complete characterization of X for eventually periodic Sturmian colorings:

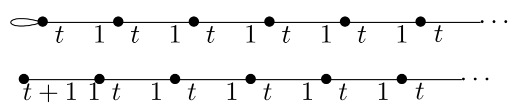
Theorem 2. *Let ϕ be a Sturmian coloring of a regular tree T.*

(1) *There exists a group Γ acting on T such that ϕ is Γ-invariant, so that ϕ is a lifting of a coloring ϕ_X on the quotient graph $X = \Gamma\backslash T$. The quotient graph $X = \Gamma\backslash T$ is one of the following two types of graphs. Here, loops are expressed by dotted lines to indicate that they may exist or not.*



(2) *If ϕ is of bounded type, then it falls into the first case above, i.e. ϕ is a lifting of a coloring of a geodesic ray with loops possibly attached.*

(3) *Moreover, ϕ is eventually periodic if and only if X is one of the following two graphs. Here the index on each oriented edge indicates the number of corresponding oriented edges in T.*



This is joint work with Seonhee Lim.

References

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