Cosmology and the Poisson summation formula

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This talk is based on:

MPT M. Marcolli, E. Pierpaoli, K. Teh, *The spectral action and cosmic topology*, arXiv:1005.2256.

The NCG standard model and cosmology

CCM A. Chamseddine, A. Connes, M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007), no. 6, 991–1089.

MP M. Marcolli, E. Pierpaoli, *Early universe models from noncommutative geometry*, arXiv:0908.3683

KM D. Kolodrubetz, M. Marcolli, *Boundary conditions of the RGE flow in noncommutative cosmology*, arXiv:1006.4000

Two topics of current interest to cosmologists:

- Modified Gravity models in cosmology:

Einstein-Hilbert action (+cosmological term) replaced or extended with other gravity terms (conformal gravity, higher derivative terms) $\Rightarrow$ cosmological predictions

- The question of Cosmic Topology:

Nontrivial (non-simply-connected) spatial sections of spacetime, homogeneous spherical or flat spaces: how can this be detected from cosmological observations?

Our approach:

- NCG provides a modified gravity model through the spectral action
- The nonperturbative form of the spectral action determines a slow-roll inflation potential
- The underlying geometry (spherical/flat) affects the shape of the potential (possible models of inflation)
- Different inflation scenarios depending on geometry
- More refined topological properties? (coupling to matter)

The noncommutative space $X \times F$ extra dimensions
product of 4-dim spacetime and finite NC space
The spectral action functional

$$\mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

$D_A = D + A + \varepsilon' J A J^{-1}$ Dirac operator with inner fluctuations
$A = A^* = \sum_k a_k[D, b_k]$

- Action functional for gravity on X (modified gravity)
- Gravity on $X \times F$ = gravity coupled to matter on X

Spectral triples $(\mathcal{A}, \mathcal{H}, D)$:

- involutive algebra $\mathcal{A}$
- representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator D on $\mathcal{H}$
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- even $\mathbb{Z}/2$-grading $[\gamma, a] = 0$ and $D\gamma = -\gamma D$
- real structure: antilinear isom $J : \mathcal{H} \to \mathcal{H}$ with $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\gamma = \varepsilon'' \gamma J$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- bimodule: $[a, b^0] = 0$ for $b^0 = Jb^* J^{-1}$
- order one condition: $[[D, a], b^0] = 0$

Asymptotic formula for the spectral action (Chamseddine–Connes)

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{k\in \mathrm{DimSp}} f_k \Lambda^k \int\!\!\!\!\!- |D|^{-k} + f(0)\zeta_D(0) + o(1)$$

for large Λ with $f_k = \int_0^\infty f(v) v^{k-1} dv$ and integration given by residues of zeta function $\zeta_D(s) = \mathrm{Tr}(|D|^{-s})$; DimSp poles of zeta functions

Asymptotic expansion $\Rightarrow$ Effective Lagrangian (modified gravity + matter)

At low energies: only nonperturbative form of the spectral action

$$\mathrm{Tr}(f(D_A/\Lambda))$$

Need explicit information on the Dirac spectrum!

Product geometry $(C^\infty(X), L^2(X,S), D_X) \cup (\mathcal{A}_F, \mathcal{H}_F, D_F)$

- $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_F = C^\infty(X, \mathcal{A}_F)$
- $\mathcal{H} = L^2(X,S) \otimes \mathcal{H}_F = L^2(X, S \otimes \mathcal{H}_F)$
- $D = D_X \otimes 1 + \gamma_5 \otimes D_F$

Inner fluctuations of the Dirac operator

$$D \to D_A = D + A + \varepsilon' J A J^{-1}$$

A self-adjoint operator

$$A = \sum a_j [D, b_j], \quad a_j, b_j \in \mathcal{A}$$

$\Rightarrow$ boson fields from inner fluctuations, fermions from $\mathcal{H}_F$

Get realistic particle physics models [CCM]
Need Ansatz for the NC space F

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

$\Rightarrow$ everything else follows by *computation*

- Representation: $\mathcal{M}_F$ sum of all inequiv irred odd $\mathcal{A}_{LR}$-bimodules (fix N generations) $\mathcal{H}_F = \oplus^N \mathcal{M}_F$ fermions
- Algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$: order one condition
- F zero dimensional but KO-dim 6
- J_F = matter/antimatter, γ_F = L/R chirality
- Classification of Dirac operators (moduli spaces)

Dirac operators and Majorana mass terms

$$D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}, \qquad S = S_1 \oplus (S_3 \otimes 1_3), \qquad T = Y_R : |\nu_R\rangle \to J_F\, |\nu_R\rangle$$

$$S_1 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 3)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 3)} \\ Y_{(\uparrow 3)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 3)} & 0 & 0 \end{pmatrix}$$

Yukawa matrices: Dirac masses and mixing angles in $\mathrm{GL}_{N=3}(\mathbb{C})$

$Y_e = Y_{(\downarrow 1)}$ (charged leptons)

$Y_\nu = Y_{(\uparrow 1)}$ (neutrinos)

$Y_d = Y_{(\downarrow 3)}$ (d/s/b quarks)

$Y_u = Y_{(\uparrow 3)}$ (u/c/t quarks)

$M = Y_R^t$ Majorana mass terms symm matrix

Moduli space of Dirac operators on finite NC space F

$$\mathcal{C}_3 \times \mathcal{C}_1$$

- $\mathcal{C}_3 =$ pairs $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo W_j unitary matrices:

$$Y'_{(\downarrow 3)} = W_1\, Y_{(\downarrow 3)}\, W_3^*\,,\;\; Y'_{(\uparrow 3)} = W_2\, Y_{(\uparrow 3)}\, W_3^*$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$: $\quad \mathcal{C}_3 = (K \times K)\backslash(G \times G)/K$
$\dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4$ (eigenval, coset 3 angles 1 phase)

- $\mathcal{C}_1 =$ triplets $(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R)$ with Y_R symmetric modulo

$$Y'_{(\downarrow 1)} = V_1\, Y_{(\downarrow 1)} V_3^*\,,\;\; Y'_{(\uparrow 1)} = V_2\, Y_{(\uparrow 1)}\, V_3^*\,,$$

$$Y'_R = V_2\, Y_R\, \bar{V}_2^*$$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection forgets Y_R fiber symm matrices mod $Y_R \mapsto \lambda^2 Y_R$
$\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$ (dim fiber 12-1=11)

Parameters of νMSM

- three coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters
- QCD vacuum angle

Moduli space of Dirac operators on F $\Rightarrow$ geometric form of all the Yukawa and Majorana parameters

Fields content of the model

- Bosons: inner fluctuations $A = \sum_j a_j[D, b_j]$
- In M direction: $U(1)$, $SU(2)$, and $SU(3)$ gauge bosons
- In F direction: Higgs field $H = \varphi_1 + \varphi_2 j$

- Fermions: basis of $\mathcal{H}_F$

$$|\uparrow\rangle \otimes \mathbf{3}^0, \quad |\downarrow\rangle \otimes \mathbf{3}^0, \quad |\uparrow\rangle \otimes \mathbf{1}^0, \quad |\downarrow\rangle \otimes \mathbf{1}^0$$

Gauge group $SU(\mathcal{A}_F) = U(1) \times SU(2) \times SU(3)$
(up to fin abelian group)

- Hypercharges: adjoint action of $\mathrm{U}(1)$ (in powers of $\lambda \in \mathrm{U}(1)$)

	$\uparrow \otimes \mathbf{1}^0$	$\downarrow \otimes \mathbf{1}^0$	$\uparrow \otimes \mathbf{3}^0$	$\downarrow \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

$\Rightarrow$ Correct hypercharges to the fermions

Action functional

$$\mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

Fermion part: antisymmetric bilinear form $\mathfrak{A}(\tilde{\xi})$ on

$$\mathcal{H}^+ = \{\xi \in \mathcal{H} \,|\, \gamma\xi = \xi\}$$

$\Rightarrow$ nonzero on Grassmann variables
Euclidean functional integral $\Rightarrow$ Pfaffian

$$Pf(\mathfrak{A}) = \int e^{-\frac{1}{2}\mathfrak{A}(\tilde{\xi})} D[\tilde{\xi}]$$

(avoids Fermion doubling problem of previous models based on symmetric $\langle \xi, D_A\xi\rangle$ for NC space with KO-dim=0)

Explicit computation gives part of SM Larangian with

- $\mathcal{L}_{Hf}$ = coupling of Higgs to fermions
- $\mathcal{L}_{gf}$ = coupling of gauge bosons to fermions
- $\mathcal{L}_{f}$ = fermion terms

The asymptotic expansion of the spectral action from [CCM]

$$
\begin{aligned}
S = \;& \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, \mathfrak{c} + \frac{f_0}{4}\, \mathfrak{d}) \int \sqrt{g}\, d^4x \\
+ \;& \frac{96\, f_2\, \Lambda^2 - f_0\, \mathfrak{c}}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0}{10\,\pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x \\
+ \;& \frac{(-2\,\mathfrak{a}\, f_2\, \Lambda^2 + \mathfrak{e}\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0 \mathfrak{a}}{2\,\pi^2} \int |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
- \;& \frac{f_0 \mathfrak{a}}{12\,\pi^2} \int R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0 \mathfrak{b}}{2\,\pi^2} \int |\varphi|^4\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0}{2\,\pi^2} \int (g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x,
\end{aligned}
$$

Parameters:

- f_0, f_2, f_4 free parameters, $f_0 = f(0)$ and, for $k > 0$,

$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ functions of Yukawa parameters of SM+r.h.ν

$$\begin{aligned}
\mathfrak{a} &= \mathrm{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)) \\
\mathfrak{b} &= \mathrm{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2) \\
\mathfrak{c} &= \mathrm{Tr}(MM^\dagger) \\
\mathfrak{d} &= \mathrm{Tr}((MM^\dagger)^2) \\
\mathfrak{e} &= \mathrm{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu).
\end{aligned}$$

Normalization and coefficients

$$\begin{aligned} S = \; & \frac{1}{2\kappa_0^2} \int R \sqrt{g}\, d^4x + \gamma_0 \int \sqrt{g}\, d^4x \\ + \; & \alpha_0 \int C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} \sqrt{g}\, d^4x + \tau_0 \int R^* R^* \sqrt{g}\, d^4x \\ + \; & \frac{1}{2} \int |DH|^2 \sqrt{g}\, d^4x - \mu_0^2 \int |H|^2 \sqrt{g}\, d^4x \\ - \; & \xi_0 \int R\, |H|^2 \sqrt{g}\, d^4x + \lambda_0 \int |H|^4 \sqrt{g}\, d^4x \\ + \; & \frac{1}{4} \int (G^i_{\mu\nu}\, G^{\mu\nu i} + F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + B_{\mu\nu}\, B^{\mu\nu}) \sqrt{g}\, d^4x, \end{aligned}$$

Energy scale: Unification ($10^{15} - 10^{17}$ GeV)

$$\frac{g^2 f_0}{2\pi^2} = \frac{1}{4}$$

Preferred energy scale, unification of coupling constants

Coefficients

$$\frac{1}{2\kappa_0^2} = \frac{96 f_2 \Lambda^2 - f_0 \mathfrak{c}}{24\pi^2} \qquad \gamma_0 = \frac{1}{\pi^2}\left(48 f_4 \Lambda^4 - f_2 \Lambda^2 \mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right)$$

$$\alpha_0 = -\frac{3 f_0}{10\pi^2} \qquad \tau_0 = \frac{11 f_0}{60\pi^2}$$

$$\mu_0^2 = 2\frac{f_2 \Lambda^2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \qquad \xi_0 = \frac{1}{12}$$

$$\lambda_0 = \frac{\pi^2 \mathfrak{b}}{2 f_0 \mathfrak{a}^2}$$

In [MP] [KM]: running coefficients with RGE flow of particle physics content from unification energy down to electroweak.
$\Rightarrow$ Very early universe models! $(10^{-36}s < t < 10^{-12}s)$

Effective gravitational constant

$$G_{\text{eff}} = \frac{\kappa_0^2}{8\pi} = \frac{3\pi}{192 f_2 \Lambda^2 - 2 f_0 \mathfrak{c}(\Lambda)}$$

Effective cosmological constant

$$\gamma_0 = \frac{1}{4\pi^2}(192 f_4 \Lambda^4 - 4 f_2 \Lambda^2 \mathfrak{c}(\Lambda) + f_0 \mathfrak{d}(\Lambda))$$

Conformal non-minimal coupling of Higgs and gravity

$$\frac{1}{16\pi G_{\text{eff}}} \int R \sqrt{g} d^4x - \frac{1}{12} \int R\, |H|^2 \sqrt{g} d^4x$$

Conformal gravity

$$\frac{-3 f_0}{10\pi^2} \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g} d^4x$$

$C^{\mu\nu\rho\sigma}$ = Weyl curvature tensor (trace free part of Riemann tensor)

Cosmological implications of the NCG SM

- Linde's hypothesis (antigravity in the early universe)
- Primordial black holes and gravitational memory
- Gravitational waves in modified gravity
- Gravity balls
- Varying effective cosmological constant
- Higgs based slow-roll inflation
- Spontaneously arising Hoyle-Narlikar in EH backgrounds

Effects in the very early universe: inflation mechanisms
- Remark: Cannot extrapolate to modern universe, nonperturbative effects in the spectral action: requires nonperturbative spectral action

Cosmological models for the not-so-early-universe?

Need to work with non-perturbative form of the spectral action

Can to for specially symmetric geometries!

Concentrate on pure gravity part: X instead of $X \times F$

The spectral action and the question of cosmic topology

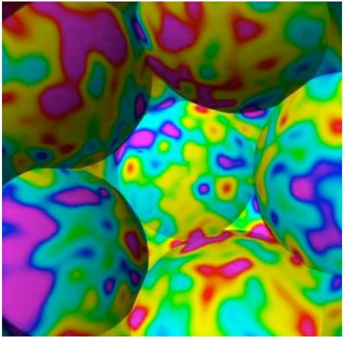
(with E. Pierpaoli and K. Teh)

Spatial sections of spacetime closed 3-manifolds $\neq S^3$?

- Cosmologists search for signatures of topology in the CMB
- Model based on NCG distinguishes cosmic topologies?

Yes! the non-perturbative spectral action predicts different models of slow-roll inflation

Cosmic topology



(Luminet, Lehoucq, Riazuelo, Weeks, et al.: simulated CMB sky)
Best candidates: Poincaré homology 3-sphere and other spherical forms (quaternionic space), flat tori
Testable Cosmological predictions? (in various gravity models)

Poisson summation formula

$$\sum_{n\in\mathbb{Z}} h(x+\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \exp\left(\frac{2\pi i n x}{\lambda}\right)\ \hat{h}(\frac{n}{\lambda})$$

$\lambda \in \mathbb{R}_+^*$ and $x \in \mathbb{R}$ with

$$\hat{h}(x) = \int_{\mathbb{R}} h(u)\, e^{-2\pi i u x}\, du$$

Idea: write $\mathrm{Tr}(f(D/\Lambda))$ as sums over lattices

- Need explicit spectrum of D with multiplicities
- Need to write as a union of arithmetic progressions $\lambda_{n,i}$, $n \in \mathbb{Z}$
- Multiplicities polynomial functions $m_{\lambda_{n,i}} = P_i(\lambda_{n,i})$

$$\mathrm{Tr}(f(D/\Lambda)) = \sum_i \sum_{n\in\mathbb{Z}} P_i(\lambda_{n,i}) f(\lambda_{n,i}/\Lambda)$$

The standard topology S^3 (Chamseddine–Connes)

Dirac spectrum $\pm a^{-1}(\frac{1}{2}+n)$ for $n \in \mathbb{Z}$, with multiplicity $n(n+1)$

$$\mathrm{Tr}(f(D/\Lambda)) = (\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{4}(\Lambda a)\widehat{f}(0) + O((\Lambda a)^{-k})$$

with $\widehat{f}^{(2)}$ Fourier transform of $v^2 f(v)$ 4-dimensional Euclidean $S^3 \times S^1$

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \pi\Lambda^4 a^3 \beta \int_0^\infty u\, h(u)\, du - \frac{1}{2}\pi\Lambda a\beta \int_0^\infty h(u)\, du + O(\Lambda^{-k})$$

$$g(u,v) = 2P(u)\, h(u^2(\Lambda a)^{-2} + v^2(\Lambda\beta)^{-2})$$

$$\widehat{g}(n,m) = \int_{\mathbb{R}^2} g(u,v) e^{-2\pi i(xu+yv)}\, du\, dv$$

A slow roll potential from non-perturbative effects
perturbation $D^2 \mapsto D^2 + \phi^2$ gives potential $V(\phi)$ scalar field coupled to gravity

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2))) = \pi\Lambda^4\beta a^3 \int_0^\infty uh(u)du - \frac{\pi}{2}\Lambda^2\beta a \int_0^\infty h(u)du$$

$$+\pi\Lambda^4\beta a^3\,\mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\,\mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))du, \qquad \mathcal{W}(x) = \int_0^x h(u)du$$

Slow roll parameters Minkowskian Friedmann metric on $S \times \mathbb{R}$

$$ds^2 = a(t)^2 ds_S^2 - dt^2$$

accelerated expansion $\frac{\ddot{a}}{a} = H^2(1-\epsilon)$ Hubble parameter

$$H^2(\phi)\left(1 - \frac{1}{3}\epsilon(\phi)\right) = \frac{8\pi}{3m_{Pl}^2} V(\phi)$$

m_{Pl} Planck mass

$$\epsilon(\phi) = \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

inflation phase $\epsilon(\phi) < 1$

$$\eta(\phi) = \frac{m_{Pl}^2}{8\pi}\left(\frac{V''(\phi)}{V(\phi)}\right) - \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

second slow-roll parameter $\Rightarrow$ measurable quantities

$$n_s = 1 - 6\epsilon + 2\eta \quad r = 16\epsilon$$

spectral index and tensor-to-scalar ratio

Slow-roll parameters from spectral action $S = S^3$

$$\epsilon(x) = \frac{m_{Pl}^2}{16\pi} \left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta(x) = \frac{m_{Pl}^2}{8\pi} \frac{h'(x) + 2\pi(\Lambda a)^2 h(x)}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}$$

$$-\frac{m_{Pl}^2}{16\pi} \left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

In Minkowskian Friedmann metric $\Lambda(t) \sim 1/a(t)$
Also independent of β (artificial Euclidean compactification)

The quaternionic space $SU(2)/Q8$ (quaternion units $\pm 1, \pm\sigma_k$)
Dirac spectrum (Ginoux)

$$\frac{3}{2} + 4k \quad \text{with multiplicity} \quad 2(k+1)(2k+1)$$

$$\frac{3}{2} + 4k + 2 \quad \text{with multiplicity} \quad 4k(k+1)$$

Polynomial interpolation of multiplicities

$$P_1(u) = \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16}$$

$$P_2(u) = \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}$$

Spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + O(\Lambda^{-k})$$

(1/8 of action for S^3) with $g_i(u) = P_i(u) f(u/\Lambda)$:

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{4}\left(\widehat{g}_1(0) + \widehat{g}_2(0)\right) + O(\Lambda^{-k})$$

from Poisson summation $\Rightarrow$ Same slow-roll parameters

The dodecahedral space Poincaré homology sphere S^3/Γ
binary icosahedral group 120 elements
Dirac spectrum: eigenvalues of S^3 different multiplicities $\Rightarrow$
generating function (Bär)

$$F_+(z) = \sum_{k=0}^{\infty} m(\frac{3}{2}+k, D) z^k \quad F_-(z) = \sum_{k=0}^{\infty} m(-(\frac{3}{2}+k), D) z^k$$

$$F_+(z) = -\frac{16(710647 + 317811\sqrt{5})G^+(z)}{(7+3\sqrt{5})^3(2207+987\sqrt{5})H^+(z)}$$

$$G^+(z) = 6z^{11}+18z^{13}+24z^{15}+12z^{17}-2z^{19}-6z^{21}-2z^{23}+2z^{25}+4z^{27}+3z^{29}+z^{31}$$

$$H^+(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
$$-9z^{22}-6z^{24}-2z^{26}+2z^{28}+4z^{30}+3z^{32}+z^{34}$$

$$F_-(z) = -\frac{1024(5374978561 + 2403763488\sqrt{5})G^-(z)}{(7+3\sqrt{5})^8(2207+987\sqrt{5})H^-(z)}$$

$$G^-(z) = 1+3z^2+4z^4+2z^6-2z^8-6z^{10}-2z^{12}+12z^{14}+24z^{16}+18z^{18}+6z^{20}$$

$$H^-(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
$$-9z^{22}-6z^{24}-2z^{26}+2z^{28}+4z^{30}+3z^{32}+z^{34}$$

Polynomial interpolation of multiplicities: 60 polynomials $P_i(u)$

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}$$

Spectral action: functions $g_j(u) = P_j(u) f(u/\Lambda)$

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60} \sum_{j=0}^{59} \widehat{g}_j(0) + O(\Lambda^{-k})$$

$$= \frac{1}{60} \int_{\mathbb{R}} \sum_j P_j(u) f(u/\Lambda) du + O(\Lambda^{-k})$$

by Poisson summation $\Rightarrow 1/120$ of action for S^3
Same slow-roll parameters

The flat tori

Dirac spectrum (Bär)

$$\pm 2\pi \parallel (m, n, p) + (m_0, n_0, p_0) \parallel, \qquad (1)$$

$(m, n, p) \in \mathbb{Z}^3$ multiplicity 1 and constant vector (m_0, n_0, p_0) depending on spin structure

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \sum_{(m,n,p)\in\mathbb{Z}^3} 2f\left(\frac{4\pi^2((m+m_0)^2 + (n+n_0)^2 + (p+p_0)^2)}{\Lambda^2}\right)$$

Poisson summation

$$\sum_{\mathbb{Z}^3} g(m, n, p) = \sum_{\mathbb{Z}^3} \widehat{g}(m, n, p)$$

$$\widehat{g}(m, n, p) = \int_{\mathbb{R}^3} g(u, v, w) e^{-2\pi i(mu+nv+pw)} dudvdw$$

$$g(m, n, p) = f\left(\frac{4\pi^2((m+m_0)^2 + (n+n_0)^2 + (p+p_0)^2)}{\Lambda^2}\right)$$

Spectral action for the flat tori

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)du\,dv\,dw + O(\Lambda^{-k})$$

$X = T^3 \times S^1_\beta$:

$$\mathrm{Tr}(h(D_X^2/\Lambda^2)) = \frac{\Lambda^4\beta\ell^3}{4\pi}\int_0^\infty uh(u)du + O(\Lambda^{-k})$$

using

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} 2\ h\left(\frac{4\pi^2}{(\Lambda\ell)^2}((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)+\frac{1}{(\Lambda\beta)^2}(r+\frac{1}{2})^2\right)$$

$$g(u,v,w,y) = 2\ h\left(\frac{4\pi^2}{\Lambda^2}(u^2+v^2+w^2)+\frac{y^2}{(\Lambda\beta)^2}\right)$$

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} g(m+m_0,n+n_0,p+p_0,r+\frac{1}{2}) = \sum_{(m,n,p,r)\in\mathbb{Z}^4}(-1)^r\,\widehat{g}(m,n,p,r)$$

Different slow-roll potential and parameters Introducing the perturbation $D^2 \mapsto D^2 + \phi^2$:

$$\mathrm{Tr}(h((D_X^2 + \phi^2)/\Lambda^2)) = \mathrm{Tr}(h(D_X^2/\Lambda^2)) + \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

slow-roll potential

$$V(\phi) = \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u\,(h(u+x) - h(u))\,du$$

Slow-roll parameters (different from spherical cases)

$$\epsilon = \frac{m_{Pl}^2}{16\pi} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta = \frac{m_{Pl}^2}{8\pi} \left(\frac{h(x)}{\int_0^\infty u(h(u+x) - h(u))du} - \frac{1}{2} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2 \right)$$

Conclusion (for now)

A modified gravity model based on the spectral action cannot rule out most likely cosmic topology candidates (dodecahedral, quaternionic) but can distinguish the spherical candidates from the flat ones on the basis of different inflation scenarios!