ON THE HAUSDORFF THEORY OF DIOPHANTINE CONDITIONS FOR A TRANSLATION SURFACE

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A *translation surface* is a genus g closed surface X with a flat metric and a finite set Σ of conical singularities $p_1, \dots, p_r$, the angle at each p_i being an integer multiple of 2π. Any translation surface admits a nice flat representation as polygon P in the complex plane, or a finite union of polygons, having an even number of sides which are grouped in pairs. Paired sides are parallel, with the same length and the same orientation, so that they can be identified by a translation, giving rise to a translation surface X. A stratum $\mathcal{H}$ is a set of translation surfaces X having the same genus g, the same number of conical singularities r and the same conical angle at any conical singularity. It is an affine orbifold with complex dimension $2g + r - 1$. Any stratum admits an action of $\mathrm{SL}(2, \mathbb{R})$. If the translation surface X is represented identifying the sides of a polygon P as above, then $A \cdot X$ is the translation surface obtained identifying the sides of the affine image $A \cdot P$ of the polygon P, with sides paired according to the same identifications as in P, indeed affine maps preserve parallelism and ratios between lengths.

We consider both parameter space and phase space dynamics. In parameter space, that is on a stratum $\mathcal{H}$, the so-called *Teichmüller flow* corresponds to the diagonal part $g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ of the action of $\mathrm{SL}(2, \mathbb{R})$ described above. In phase space, that is on a fixed translation surface X, the *linear flow* ϕ^t_θ in a fixed direction θ is the flow that, in a flat representation of X by polygons, corresponds to move at constant speed in direction θ and then following the identification at the boundary of P.

We fix a translation surface X. In strata, we are interested to the dependence on θ of the asymptotic behaviour of the orbit $g_t r_\theta \cdot X$ for $t > 0$, where $r_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. On the surface X itself we are interested on the dependence on θ of the recurrence properties of the linear flow ϕ^t_θ. In both cases, for us the parameter space is the interval $(-\pi/2, \pi/2)$, and we prove estimates on the *Hausdorff f-measure* H^f of the set of those directions θ such that either the generic orbit of ϕ^t_θ has a prescribed recurrence, or $g_t r_\theta \cdot X$ makes excursions at infinity at prescribed rate. Here f is a *dimension function*, that is a continuous increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(r)/r$ is decreasing with $\lim_{r \to 0} f(r)/r = \infty$. We also admit the classical Lebesgue measure, corresponding to the dimension function $f(r) = r$. When $f(r) = r^s$ with $0 < s \leq 1$, the measure H^f is the usual s-dimensional Hausdorff measure H^s, which gives rise to the classical notion of *Hausdorff dimension* dim. Different dimension functions, like for example $f(r) := \log\left(r^{-1}\right)^{1/w} \log\log\left(r^{-1}\right)^{-1}$, where $w > 0$ is a positive parameter, give rise to the so-called *logarithmic dimension* $\dim_{\mathcal{L}}$.

1. Excursions of Teichmüller geodesics

A *saddle connection* for X is a segment γ of geodesic for the flat metric connecting two conical singularities p_i and p_j and not containing other conical singularities in its interior. The *systole* $\mathrm{Sys}(X)$ of X is the length $|\gamma|$ of the shortest saddle connection γ for X. According to the so-called *Malher criterion* the compacts subsets of a stratum are subsets $\mathcal{K}_\epsilon$ such that $\mathrm{Sys}(X) \geq \epsilon$ for any $X \in \mathcal{K}_\epsilon$ and for some fixed $\epsilon > 0$. Fix a surface X with $\mathrm{Sys}(X) > \epsilon$. The directions $\theta \in (-\pi/2, \pi/2)$ giving rise to positive geodesics eventually contained in $\mathcal{K}_\epsilon$ are the element of the set

$$\mathrm{Bad}^{sc}(X, \epsilon) := \{\theta \ ; \ \mathrm{Sys}(g_t r_\theta X) \geq \epsilon \text{ for all } t \text{ big enough }\}.$$

Theorem 1.1 below is a generalization of *Jarnick's inequality.* The Theorem is also a quantitative version of a previous result of Kleinbock, Weiss on *thickness* of the set of θ giving rise to a bounded geodetic. Theorem 1.1 is a consequence of Theorem 3.3 in the following.

Theorem 1.1. *Let X be a translation surface. Then there exist positive constants ϵ_0, α, c_u and c_l, depending on X, such that for $0<\epsilon<\epsilon_0$ we have*

$$1-c_l\cdot\frac{\epsilon^{2/3\alpha}}{|\log(\epsilon)|}\leq\dim\left(\mathrm{Bad}^{sc}(X,\epsilon)\right)\leq 1-c_u\cdot\frac{\epsilon^2}{|\log(\epsilon)|}. \tag{1.1}$$

Finer estimations can be proved for the Hausdorff measure H^f of the set of geodesics having unbounded excursions to the non compact part of strata. Such estimates follow from an arithmetic statement, namely Theorem 3.2 in the following. Unfortunately, while Theorem 3.2 admits a very general statement, its dynamical consequences cannot be explicitly stated in full generality. Theorem 1.2 below is a variation on a result of one of the authors (see [Mar]).

Theorem 1.2. *Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a decreasing monotone function.*

(1) *If $\int_1^\infty\varphi(t)dt=+\infty$ then for almost any θ we have* $\liminf_{t\to\infty}\dfrac{\mathrm{Sys}(g_tr_\theta X)}{\sqrt{\varphi(t)}}=0.$

(2) *If $\int_1^\infty\varphi(t)dt<+\infty$ then for almost any θ we have* $\lim_{t\to\infty}\dfrac{\mathrm{Sys}(g_tr_\theta X)}{\sqrt{\varphi(t)}}=+\infty.$

In particular, for the one parameter family of functions $\varphi_\epsilon(t):=t^{-(1+\epsilon)}$, applying both parts of the Theorem, we have

$$\limsup_{t\to\infty}\frac{-\log\mathrm{Sys}(g_tr_\theta\cdot X)}{\log t}=\frac{1}{2}\text{ for almost every }\theta. \tag{1.2}$$

Equation (1.2) above gives the maximal asymptotic excursion of the geodesic in the generic direction. Different asymptotic rates for the maximal excursions have zero Lebesgue measure, but they can be measured by generalized Hausdorff measures H^f via Theorem 3.2 below. The directions generating excursion at *diophantine rate* are the direction $\theta\in(-\pi/2,\pi/2)$ in the set

$$S_X(\alpha):=\left\{\theta\ ;\ \limsup_{t\to\infty}\frac{\log\mathrm{Sys}(g_tr_\theta\cdot X)-\alpha t}{\log t}=1\right\},$$

where α is a fixed real number with $0<\alpha<1$. For a fixed real number with $w>0$, the directions generating excursion at *Liouville rate* are the direction $\theta\in(-\pi/2,\pi/2)$ in the set

$$\mathcal{L}_X(\omega)=\left\{\theta\ ;\ \limsup_{t\to\infty}\frac{\log\mathrm{Sys}(g_tr_\theta\cdot X)-(t-\omega\log(t))}{\log t}=\frac{1}{2}\right\}.$$

Theorem 1.3 (Logarithmic Laws)**.** *Let X be any translation surface.*

(1) *For any $\alpha\in(0,1)$ we have $\dim\left(S_X(\alpha)\right)=1-\alpha$, where* dim *denotes the Hausdorff dimension.*

(2) *For any $\omega>0$ we have $\dim_{\mathcal{L}}\left(\mathcal{L}_X(\omega)\right)=\dfrac{1}{\omega}$, where $\dim_{\mathcal{L}}$ denotes the logarithmic dimension.*

2. Fast recurrence for polygonal billiards and translation flows

Let Q be a *rational polygon*, that is a polygon in the plane whose angles are rational multiples of π. The reflections at the corners generate a finite group Γ, so that directions θ in Q are decomposed into equivalence classes $[\theta]$ under the action of Γ. For any class of directions $[\theta]$ it is well defined the *billiard flow* $\widehat{\phi}_{[\theta]}$ in the class of directions $[\theta]$, that we will simply refer to as the billiard flow in direction θ. A classical *unfolding construction* defines a translation surface X with a directional flow ϕ_θ^t starting from Q and the class $[\theta]$. Understanding the billiard flow on rational polygons is one of the motivations for introducing and studying translation surfaces. In particular, by a theorem of Masur, the Hausdorff dimension $\lambda=\lambda(Q)$ of the set of direction on Q such that $\widehat{\phi}_{[\theta]}$ is not uniquely ergodic satisfies $0\leq\lambda\leq1/2$. When θ is an uniquely ergodic direction one can define a flow-invariant and thus almost everywhere constant function

$$\tau_\theta:Q\to\mathbb{R}_+\quad;\quad\tau_\theta(p):=\liminf_{r\to0}\frac{\log\left(\tau_\theta(p,r)\right)}{-\log r},$$

where for any $r > 0$ the quantity $\tau_\theta(p,r) := \inf\{t > r; |\widehat{\phi}_{[\theta]}(p) - p| < r\}$ denotes the *return time* of p at scale r. The function τ_θ determines how return times grow asymptotically, and it is proved by D. H. Kim and S. Marmi that $\tau_\theta(p) = 1$ for almost any θ and almost any p. Theorem 2.1 below was obtained simultaneously in this paper and in a paper of L. Marchese with D. H. Kim and S. Marmi, and considers the Hausdorff dimension of directions having fast recurrence.

Theorem 2.1. *Let Q be any rational polygon and let $0 \leq \lambda \leq 1/2$ be the dimension of the set of non uniquely ergodic directions on Q. Then for any η with $2 < \eta < 2/\lambda$ there exists a set of directions S_η with $\dim\big(S_\eta\big) = \dfrac{2}{\eta}$ such that for any $\theta \in S_\eta$ and almost any $p \in Q$ we have*

$$\tau_\theta(p) = \frac{1}{\eta - 1}.$$

3. Diophantine approximations and planar resonant sets

We consider diophantine conditions in terms of approximation of a given direction in the euclidian plane by the directions of a countable set of points, where the quality of the approximation is measured in terms of the distance from the origin. Such approach is naturally formalized in polar coordinates, via the notion of *planar resonant set* below. For simplicity, parametrize the angular coordinate in the plane by the length coordinate in the interval $[0,1]$ and denote $B(\theta, r)$ the open subinterval of $[0,1]$ with length $2r$ centered at θ. For any measurable subset $S \subset [0,1]$ we denote $|E|$ its Lebesgue measure.

A *planar resonant set* is the datum $(\mathcal{R}, l)$, where $\mathcal{R}$ is a countable subset $\mathcal{R} \subset [0,1]$ of *resonant points* and $l : \mathcal{R} \to \mathbb{R}_+$ is a positive function, such that for any $R > 0$ the set $\{\theta \in \mathcal{R}\ ;\ l(\theta) < R\}$ is finite. An *approximation function* is a decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. The set of points in $[0,1]$ which are *well approximable* by resonant points in $\mathcal{R}$ with respect to ψ is

$$W(\mathcal{R}, \phi) := \bigcap_{R>0} \bigcup_{l(\theta)>R} B(\theta, \psi(l(\theta))).$$

Given $\epsilon > 0$, the set of points in $[0,1]$ which are *ϵ-badly approximable* with respect to $\mathcal{R}$ is

$$\mathrm{Bad}(\mathcal{R}, \epsilon) := [0,1] \setminus \bigcap_{N=1}^{\infty} \bigcup_{l(\theta)>N} B\big(\theta, \frac{\epsilon^2}{l(\theta)^2}\big).$$

Definition 3.1. *Let $(\mathcal{R}, l)$ be a planar resonant set.*

The set $(\mathcal{R}, l)$ has quadratic growth *if there exists a constant $M > 0$ such that for any $R > 0$ we have*

$$\sharp\{\theta \in \mathcal{R}\ ;\ l(\theta) < R\} < M \cdot R^2. \tag{3.1}$$

The set $(\mathcal{R}, l)$ has isotropic quadratic growth *if there exists a constant $M > 0$ such that for any subinterval $I \subset [0,1]$ and any $R > 0$ such that $R^2|I| \geq 1$ we have*

$$\sharp\{\theta \in I \cap \mathcal{R}\ ;\ l(\theta) < R\} < M \cdot |I| \cdot R^2. \tag{3.2}$$

The set $(\mathcal{R}, l)$ is ubiquitous*, or extensively (K, c_1, c_2, a)*-ubiquitous *if there exists a positive integer $K \geq 2$ and positive constant $c_1 > 0$, $c_2 > 0$ and $a > 0$ such that for any n and any interval $I \subset [0,1]$ such that*

$$|I| > \frac{c_2}{K^n}$$

we have

$$\left| I \cap \bigcup_{j=1}^{n} \bigcup_{K^{j-1} \leq l(\theta) < K^j} B\left(\theta, \frac{a}{K^{2n}}\right) \right| \geq c_1 |I|. \tag{3.3}$$

The set $(\mathcal{R}, l)$ satisfies the (ϵ, U, τ)-Dirichlet's property *for $\epsilon > 0$, $U \geq 0$ and $1 < \tau < 0$ if there exist some $L_0 > 0$ such that for any $L > L_0$ and any interval $I \subset [0,1]$ with $|I| = 2U/L^2$ we have*

$$\left| I \cap \bigcup_{l(\theta)<L} B\left(\theta, \frac{\epsilon^2}{2l(\theta)^2}\right) \right| \geq \tau |I|. \tag{3.4}$$

The set $(\mathcal{R},l)$ *is* (ϵ,K,τ)-decaying, *if there exist positive constants* $\epsilon>0$ *and* $0<\tau<1$, *an integer* $K\geq 1/\epsilon$ *and an integer* N *such that the following holds. For any* $n\geq N$ *and any sub-interval* $I\subset[0,1]$ *with*

$$|I|=\frac{1}{K^{2n}} \qquad \textit{and} \qquad I\cap\bigcup_{j=1}^{n-1}\bigcup_{K^{j-1}\leq l(\theta)<K^{j}} B\left(\theta,\frac{\epsilon^2}{l(\theta)\cdot K^j}\right)=\emptyset \tag{3.5}$$

we have

$$\left|I\cap\bigcup_{K^{n-1}\leq l(\theta)<K^{n}} B\left(\theta,\frac{2\epsilon^2}{l(\theta)\cdot K^n}\right)\right|\leq\tau\cdot|I|. \tag{3.6}$$

Moreover there exists an interval $I_0\subset[0,1]$ *satisfying Condition* (3.5) *for* $n=N$.

Theorem 3.2 below is a generalization of classical Khinchin Theorem to the setting of Hausdorff theory. It is an adaptation to planar resonant sets of well known results of Beresnevich and Velani ([Be,Ve]). Related versions of the same statement appear also in [Bo,Ch].

Theorem 3.2 (Khinchin-Jarnik)**.** *Consider a planar resonant set* $(\mathcal{R},l)$, *an approximation function* ψ *and a dimension function* f *such that the composition* $f\circ\psi$ *is decreasing monotone.*

(1) *If* $\int_0^\infty rf\circ\psi(r)$ *converges at* $r=+\infty$ *then we have* $H^f\big(W(\mathcal{R},\psi)\big)=0$.

(2) *If* $\int_0^\infty rf\circ\psi(n)$ *diverges at* $r=+\infty$ *and moreover* $(\mathcal{R},l)$ *is ubiquitous and has isotropic quadratic growth, then we have* $H^f\big(W(\mathcal{R},\psi)\big)=H^f\big([0,1]\big)$.

Theorem 3.3 is a generalization of Jarnik's inequality.

Theorem 3.3 (Abstract Jarnik's inequality)**.** *Consider a planar resonant set* $(\mathcal{R},l)$.

(1) *If* $(\mathcal{R},l)$ *is* (ϵ,K,τ)*-decaying then we have*

$$\dim\left(\mathrm{Bad}\left(\epsilon/\sqrt{K}\right)\right)\geq 1-\frac{|\log(1-\tau)|}{\log(K^2)}.$$

(2) *If* $(\mathcal{R},l)$ *satisfies the* (ϵ,U,τ)-Dirichlet property *for* $\epsilon>0$, $U\geq 0$ *and* $1<\tau<0$

$$\dim\big(\mathrm{Bad}(\epsilon)\big)\leq 1-\frac{|\log(1-\tau)|}{-\log(\epsilon^2/(8U))}.$$

4. Holonomy resonant sets for translation surfaces

Fix a translation surface X. If γ is a saddle connection for X, denote $\theta(\gamma)$ its direction. It is well known that for a given direction θ there exist at most $4g-4$ saddle connections γ_i such that $\theta(\gamma_i)=\theta$ for any i. For a direction $\theta=\theta(\gamma)$ of a saddle connection γ let $\gamma^{max}(\theta)$ be the saddle connection parallel to γ with maximal length. Define the planar resonant set $\mathcal{R}^{sc}$ and the length function $l:\mathcal{R}^{sc}\to\mathbb{R}_+$ by

$$\mathcal{R}^{sc}:=\{\theta=\theta(\gamma)\ ;\ \gamma \text{ saddle connection for } X\}$$
$$l(\theta):=|\gamma^{max}(\theta)|.$$

We consider also *closed geodesics* σ for X. For any such σ there exists a family of closed geodesics which are parallel to σ with the same length and the same orientation. A *cylinder* is a connected open set C_σ foliated by such a family of parallel closed geodesics and maximal with this property. By maximality, it follows that the boundary of a cylinder C_σ around a closed geodesic σ is union of saddle connections parallel to σ. For a direction $\theta=\theta(\sigma)$ of a closed geodesic σ let $\sigma^{max}(\theta)$ be the closed geodesic parallel to σ with maximal length. Introduce the constant

$$a(X):=\frac{\mathrm{Sys}(X)^2}{2^{2^{2g-2}}}.$$

Define the planar resonant set $\mathcal{R}^{cyl}$ and the length function $l:\mathcal{R}^{cyl}\to\mathbb{R}_+$ by

$$\mathcal{R}^{cyl}:=\{\theta=\theta(\sigma)\ ;\ \gamma \text{ closed geodesic for } X \text{ with } \mathrm{Area}(C_\sigma)>a(X)\}$$
$$l(\theta):=|\sigma^{max}(\theta)|.$$

The quadratic growth for resonant sets arising from translation surfaces is established by a well-known theorem of Masur.

Theorem 4.1 (Masur)**.** *For any translation surface the planar resonant sets $\mathcal{R}^{sc}$ and $\mathcal{R}^{cyl}$ have quadratic growth.*

In this paper, collecting previous results of Vorobets, Chaika, Minsky-Weiss, we prove the other properties of holonomy resonant sets.

Theorem 4.2. *Let X be a translation surface. There exists constants $\kappa > 1$ and $\alpha > 0$, depending only on the stratum $\mathcal{H}$ of X such that the following holds.*

(1) *The set $\mathcal{R}^{sc}$ is (ϵ, K, τ)-decaying with*

$$K = \left[\frac{1}{\epsilon}\right] + 1 \qquad \textit{and} \qquad \tau = \kappa\epsilon^{\alpha},$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Moreover $\mathcal{R}^{sc}$ satisfies (U, ϵ, τ)-Dirichlet property with

$$U = \frac{\kappa^2}{\epsilon^2} \qquad \textit{and} \qquad \tau = \frac{\epsilon^4}{8\kappa^2}.$$

(2) *The set $\mathcal{R}^{cyl}$ has isotropic quadratic growth and is (K, c_1, c_2, a)-ubiquitous for proper constants K, c_1, c_2, a which depend only on the systole* $\mathrm{Sys}(X)$ *of X.*

It is possible to see that for a generic translation surface X the set $\mathcal{R}^{sc}$ does not have isotropic quadratic growth.

4.1. **Consequences.** From the results presented above on can easily deduce the following extra results.

(1) Since the boundary of a cylinder around a closed geodesic σ is union of saddle connections γ_i, with $|\gamma_i| \leq |\sigma|$, then $\mathcal{R}^{sc}$ satisfies ubiquity. For the same reason the resonant set $\mathcal{R}^{cyl}$ is decaying and satisfies Dirichlet's property wit the same constants as $\mathcal{R}^{sc}$.
(2) The sets $\mathcal{R}^{cyl}$ and $\mathcal{R}^{sc}$ satisfy the dichotomy in Theorem 3.2.
(3) Combining the results in Theorem 4.2 and in Theorem 1.1 one that there exists constants $\kappa > 1$ and $\alpha > 0$, depending only on the stratum $\mathcal{H}$ of X, such that both for $\mathcal{R} = \mathcal{R}^{cyl}$ and for $\mathcal{R} = \mathcal{R}^{cyl}$, and for any $\epsilon > 0$ small enough we have

$$1 - \kappa\frac{\epsilon^{2\alpha}}{|\log\epsilon|} < \dim\big(\mathrm{Bad}(\mathcal{R}, \epsilon)\big) < 1 - \frac{1}{\kappa}\frac{\epsilon^4}{|\log\epsilon|}.$$

References

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