Quantum Field Theory, Topology and Duality

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Motivation

In order to construct M-theory/String Theory axiomatically, we need to know

- Degrees of freedom: Perturbative string spectrum (graviton, gauge fields), D-branes (K-theory, gerbes)
- Symmetries: Besides usual general coordinate invariance, gauge symmetry, . . . , String Theory possesses additional discrete symmetries (T-duality, S-duality, . . .).

For a QFT on a manifold M the degrees of freedom are locally (i.e. on a coordinate patch $U_\alpha \subset M$) given in terms of fields $g_{\mu\nu}, A_\mu, B_{\mu\nu}, \ldots$, related in overlaps by coordinate transformations, gauge transformations, etc.

Globally, we can think of these fields as sections of bundles over M, connections on vector bundles, gerbe connections, ...

In String Theory we may also allow for the fields in overlaps to be related by the additional discrete symmetries (T-duality, S-duality, ...). In that case the underlying manifold is no longer geometric. What is the global meaning?

There are two proposals (based on considering examples of T-duality)

- Noncommutative geometry (open strings): The "nongeometric manifolds" are continuous fields of noncommutative tori over a base manifold M (price to pay: not locally trivial).
- Generalized geometry (closed strings): The "nongeometric manifolds" are so-called T-folds (price to pay: doubling of the dimensions) [Hull].

In this talk I will review the global aspects of T-duality and show why Generalized Geometry, as introduced by Hitchin, provides a natural framework to discuss T-duality.
If time allows, I will also show how generalized (complex) geometries show up naturally in the study of two-dimensional sigma models.

References

- P Bouwknegt, J Evslin and V Mathai, [hep-th/0306062,0312052]
- P Bouwknegt, K Hannabuss and V Mathai, [hep-th/0312284,0412092,0412268]
- V Mathai and J Rosenberg, [hep-th/0401168,0409073,0508084]
- U Bunke, (P Rumpf) and T Schick, [math.GT/0405132,0501487]
- C Hull, [hep-th/0406102]
- P Bouwknegt, J. Garretson and P Kao, to appear

Also relying heavily on earlier papers by, in particular, Hitchin, Cavalcanti, Gualtieri, ...
Klimčík, Strobl, Schaller, Alekseev, Cattaneo, Felder, Park, Hofman, Stojevich, Halmagyi, ...

Closed string on $M \times S^1$

Closed strings on $M \times S^1$ are described by

$$X : \Sigma \rightarrow M \times S^1$$

where $\Sigma = \{(\sigma, \tau)\}$ is the closed string worldsheet.
Upon quantization, we find

- Momentum modes: $p = \frac{n}{R}$
- Winding modes: $X(0, \tau) \sim X(1, \tau) + mR$

$$\text{Mass}^2 = \left(\frac{n}{R}\right)^2 + (mR)^2 + \text{osc. modes}$$

We have a duality $R \rightarrow 1/R$, such that ST on $M \times S^1$ is equivalent to ST on $M \times \widehat{S}^1$ (or a duality between IIA and IIB ST, for susy ST)

The Buscher rules

Low energy effective action given by (conformally invariant) σ-model

$$S = \int \Big[\sqrt{h}h^{\alpha\beta} g_{MN}(X)\partial_\alpha X^M \partial_\beta X^N + \epsilon^{\alpha\beta} B_{MN}(X)\partial_\alpha X^M \partial_\beta X^N + \sqrt{h}R(h)\Phi(X)\Big]$$

Now, suppose we have a $U(1)^N$ isometry $X^m \to X^m + \epsilon^m$, then this action has a symmetry given by the Buscher rules

$$\begin{aligned}\widehat{Q}_{MN} &= \begin{pmatrix} \widehat{Q}_{\mu\nu} & \widehat{Q}_{\mu n} \\ \widehat{Q}_{m\nu} & \widehat{Q}_{mn} \end{pmatrix} \\ &= \begin{pmatrix} Q_{\mu\nu} - Q_{\mu m}(Q^{-1})^{mn}Q_{n\nu} & -Q_{\mu m}(Q^{-1})^{m}{}_{n} \\ (Q^{-1})_{m}{}^{n}Q_{n\nu} & (Q^{-1})_{mn} \end{pmatrix}\end{aligned}$$

More explicitly, for a $U(1)$ isometry,

$$
\begin{aligned}
\widehat{g}_{\bullet\bullet} &= \frac{1}{g_{\bullet\bullet}} \\
\widehat{g}_{\bullet\mu} &= \frac{B_{\bullet\mu}}{g_{\bullet\bullet}} \\
\widehat{g}_{\mu\nu} &= g_{\mu\nu} - \frac{1}{g_{\bullet\bullet}} \left(g_{\bullet\mu} g_{\bullet\nu} - B_{\bullet\mu} B_{\bullet\nu} \right) \\
\widehat{B}_{\bullet\mu} &= \frac{g_{\bullet\mu}}{g_{\bullet\bullet}} \\
\widehat{B}_{\mu\nu} &= B_{\mu\nu} - \frac{1}{g_{\bullet\bullet}} \left(g_{\bullet\mu} B_{\bullet\nu} - g_{\bullet\nu} B_{\bullet\mu} \right)
\end{aligned}
$$

Principal S^1-bundles

Suppose we have a pair (E, H), consisting of a principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

and a so-called H-flux H, a Čech 3-cocycle.

Topologically, E is classified by an element in $F \in H^2(M, \mathbb{Z})$ while H gives a class in $H^3(E, \mathbb{Z})$

Result

The T-dual of (E, H) is given by the pair $(\widehat{E}, \widehat{H})$, where the principal S^1-bundle

$$\begin{array}{ccc} \widehat{S}^1 & \longrightarrow & \widehat{E} \\ & & \Big\downarrow{\scriptstyle \widehat{\pi}} \\ & & M \end{array}$$

and the dual H-flux $\widehat{H} \in H^3(\widehat{E}, \mathbb{Z})$, satisfy

$$\boxed{\widehat{F} = \pi_* H\,, \qquad F = \widehat{\pi}_* \widehat{H}}$$

where $\pi_* : H^3(E, \mathbb{Z}) \to H^2(M, \mathbb{Z})$, and $\widehat{\pi}_* : H^3(\widehat{E}, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ are the pushforward maps ('integration over the S^1-fiber')

$$
\begin{tikzcd}
 & E \times_M \widehat{E} \arrow[dl, "p=1\times\widehat{\pi}"] \arrow[dr, "\widehat{p}=\pi\times 1"'] & \\
E \arrow[dr, "\pi"'] & & \widehat{E} \arrow[dl, "\widehat{\pi}"] \\
 & M &
\end{tikzcd}
$$

The ambiguity in the choice of $\widehat{H}$ is removed by requiring that

$$p^* H - \widehat{p}^* \widehat{H} \equiv 0$$

in $H^3(E \times_M \widehat{E}, \mathbb{Z})$, where $E \times_M \widehat{E}$ is the correspondence space

$$E \times_M \widehat{E} = \{(x, \widehat{x}) \in E \times \widehat{E} \mid \pi(x) = \widehat{\pi}(\widehat{x})\}$$

Locally, the transformation rules on the massless low-energy effective fields (g_{MN}, B_{MN}) are consistent with the Buscher rules.

In particular, since we claim that (type IIA/B) String Theory on E, in the presence of a background H-flux H, is T-dual to (type IIB/A) String Theory on $\widehat{E}$, with background H-flux $\widehat{H}$, the spectrum of D-branes should coincide.

Theorem: This T-duality gives rise to an isomorphism between the twisted K-theories of (E, H) and $(\widehat{E}, \widehat{H})$
(with a shift in degree by 1)

- T-duality exchanges momentum (related to TE), with winding (related to T^*E) of a string.

  A natural geometric framework for T-duality is therefore a framework which treats TE and T^*E on equal footing.

$$\Downarrow$$

GENERALIZED GEOMETRY

Generalized geometry

Replace structures on TE by structures on $TE \oplus T^*E$

- Bilinear form on sections $(X, \Xi) \in \Gamma(TE \oplus T^*E)$

$$\langle (X_1, \Xi_1), (X_2, \Xi_2) \rangle = \tfrac{1}{2}(\imath_{X_1}\Xi_2 + \imath_{X_2}\Xi_1)$$

- (twisted) Courant bracket

$$[\![\, (X_1, \Xi_1), (X_2, \Xi_2) \,]\!]_H =$$
$$([X_1, X_2], \mathcal{L}_{X_1}\Xi_2 - \mathcal{L}_{X_2}\Xi_1 - \tfrac{1}{2}d\left(\imath_{X_1}\Xi_2 - \imath_{X_2}\Xi_1\right) + \imath_{X_1}\imath_{X_2}H)$$

where $H \in \Omega^3_{\mathrm{cl}}(E)$

Generalized Geometry (cont'd)

- Clifford algebra

$$\{\gamma_{(X_1,\Xi_1)},\gamma_{(X_2,\Xi_2)}\} = 2\langle (X_1,\Xi_1),(X_2,\Xi_2)\rangle$$

- Clifford module $\Omega^\bullet(E)$

$$\gamma_{(X,\Xi)}\cdot\Omega = \imath_X\Omega + \Xi\wedge\Omega$$

- (twisted) Differential on $\Omega^\bullet(E)$

$$d_H\Omega = d\Omega + H\wedge\Omega$$

Properties of the Courant bracket

For $A, B, C \in \Gamma(TE \oplus T^*E)$, $f \in C^\infty(E)$,

(a)

$$[\![A, B]\!] = -[\![B, A]\!]$$

(b)

$$\mathrm{Jac}(A, B, C) = [\![[\![A, B]\!], C]\!] + \mathrm{cycl} = d\mathrm{Nij}(A, B, C)$$

with

$$\mathrm{Nij}(A, B, C) = \tfrac{1}{3}\left(\langle [\![A, B]\!], C\rangle + \mathrm{cycl}\right)$$

(c)

$$[\![A, fB]\!] = f[\![A, B]\!] + (\rho(A)f)B - \langle A, B\rangle df$$

where $\rho : TE \oplus T^*E \to TE$ is the projection.

[Note that isotropic, involutive subbundles $A \subset TE \oplus T^*E$ (Dirac structures) give rise to Lie algebroids.]

(d) Symmetries of $\langle\cdot,\cdot\rangle$ are given by orthogonal group $O(TM \oplus T^*M) \cong O(d,d)$.
A particular kind of orthogonal transformation is the so-called B-field transform. For $b \in \Omega^2(E)$

$$e^b(X,\Xi) = (X, \Xi + \imath_X b)$$

We have

$$e^b[\![A,B]\!]_H = [\![e^bA, e^bB]\!]_{H+db}$$

Courant bracket as a derived bracket

We have the following 'Cartan formulas'

$$\begin{aligned}
\{\gamma_{(X_1,\Xi_1)}, \gamma_{(X_2,\Xi_2)}\} &= 2\langle (X_1,\Xi_1),(X_2,\Xi_2)\rangle \\
\{d_H, \gamma_{(X,\Xi)}\} &= \mathcal{L}_{(X,\Xi)} \\
[\mathcal{L}_{(X_1,\Xi_1)}, \gamma_{(X_2,\Xi_2)}] &= \gamma_{(X_1,\Xi_1)\circ(X_2,\Xi_2)} \\
[\mathcal{L}_{(X_1,\Xi_1)}, \mathcal{L}_{(X_2,\Xi_2)}] &= \mathcal{L}_{(X_1,\Xi_1)\circ(X_2,\Xi_2)} = \mathcal{L}_{[\![(X_1,\Xi_1),(X_2,\Xi_2)]\!]}
\end{aligned}$$

where

$$\mathcal{L}_{(X,\Xi)} \cdot \Omega = \mathcal{L}_X \Omega + (d\Xi + \imath_X H) \wedge \Omega$$

and the Dorfmann bracket is defined by

$$(X_1,\Xi_1)\circ(X_2,\Xi_2) = ([X_1,X_2], \mathcal{L}_{X_1}\Xi_2 - \imath_{X_2} d\Xi_1 + \imath_{X_1}\imath_{X_2} H)$$

T-duality for principal circle bundles

Given a principal circle bundle E with H-flux $H \in \Omega^3_{\rm cl}(E)_{S^1}$

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & M \end{array} \qquad H = H_{(3)} + A \wedge H_{(2)}, \; F = dA$$

there exists a T-dual principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & \widehat{E} \\ & & \Big\downarrow{\scriptstyle \widehat{\pi}} \\ & & M \end{array} \qquad \widehat{H} = H_{(3)} + \widehat{A} \wedge F, \; \widehat{F} = H_{(2)} = d\widehat{A}$$

Theorem [Bouwknegt-Evslin-Mathai, Cavalcanti-Gualtieri]

(a) We have an isomorphism of differential complexes $\tau : (\Omega^\bullet(E)_{S^1}, d_H) \to (\Omega^\bullet(\widehat{E})_{S^1}, d_{\widehat{H}})$

$$\tau(\Omega_{(k)} + A \wedge \Omega_{(k-1)}) = -\Omega_{(k-1)} + \widehat{A} \wedge \Omega_{(k)}$$
$$\tau \circ d_H = -d_{\widehat{H}} \circ \tau$$

Hence, τ induces an isomorphism on twisted cohomology

(b) We can identify $(X, \Xi) \in \Gamma(TE \oplus T^*E)_{S^1}$ with a quadruple $(x, f; \xi, g)$

$$X = x + f\partial_A\,, \qquad \Xi = \xi + gA$$

and define a map $\phi : \Gamma(TE \oplus T^*E)_{S^1} \to \Gamma(T\widehat{E} \oplus T^*\widehat{E})_{S^1}$

$$\phi(x + f\partial_A + \xi + gA) = x + g\partial_{\widehat{A}} + \xi + f\widehat{A}$$

The map ϕ is orthogonal wrt pairing on $TE \oplus T^*E$, hence τ induces an isomorphism of Clifford algebras

Theorem (cont'd)

(c) For $(X, \Xi) \in \Gamma((TE \oplus T^*E)_{S^1})$ we have

$$\tau(\gamma_{(X,\Xi)} \cdot \Omega) = \gamma_{\phi(X,\Xi)} \cdot \tau(\Omega)$$

Hence τ induces an isomorphism of Clifford modules

(d) For $(X_i, \Xi_i) \in \Gamma((TE \oplus T^*E)_{S^1})$ we have

$$\phi([\![(X_1, \Xi_1), (X_2, \Xi_2)]\!]_H) = [\![\phi(X_1, \Xi_1), \phi(X_2, \Xi_2)]\!]_{\widehat{H}}$$

Hence ϕ gives a homomorphism of twisted Courant brackets

It follows that T-duality acts naturally on generalized complex structures, generalized Kähler structures, generalized Calabi-Yau structures, ...

(e) Generalized metric on $TE \oplus T^*E$

$$\mathcal{G} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

Note that $\mathcal{G}^2 = 1$. We have

$$\begin{aligned} C_+ &= \mathrm{Ker}(\mathcal{G} - 1) = \{(X, (g+b)(X)),\ X \in \Gamma(TE)\} \\ &= \mathrm{graph}(g + b : TE \to T^*E)\,, \end{aligned}$$

The transformed generalized metric $\widehat{\mathcal{G}}$ is given by

$$\widehat{C}_+ = \mathrm{graph}(\widehat{g} + \widehat{b} : T\widehat{E} \to T^*\widehat{E})$$

where $(\widehat{g}, \widehat{b})$ are given by the Buscher rules.

Proof (sketch)

We have

$$d_H = \bar{d} + H_{(3)} + F\partial_A + A \wedge H_{(2)}$$

which proves

$$\tau \circ d_H = -d_{\widehat{H}} \circ \tau$$

The isomorphism of Clifford algebra and modules follows just as easily, and the statement on the Courant bracket follows from the Cartan formulas. $\square$

Dimensionally reduced Courant bracket

$$
\begin{aligned}
&[\![(x_1, f_1; \xi_1, g_1), (x_2, f_2; \xi_2, g_2)]\!]_{F,H} = \\
&\qquad ([x_1, x_2], x_1(f_2) - x_2(f_1) + \imath_{x_1}\imath_{x_2}F; \\
&\qquad (\mathcal{L}_{x_1}\xi_2 - \mathcal{L}_{x_2}\xi_1) - \tfrac{1}{2}d(\imath_{x_1}\xi_2 - \imath_{x_2}\xi_1) + \imath_{x_1}\imath_{x_2}H_{(3)} \\
&\qquad + \tfrac{1}{2}(df_1 g_2 + f_2 dg_1 - f_1 dg_2 - df_2 g_1) \\
&\qquad + (g_2\imath_{x_1}F - g_1\imath_{x_2}F) + (f_2\imath_{x_1}H_{(2)} - f_1\imath_{x_2}H_{(2)}), \\
&\qquad\quad x_1(g_2) - x_2(g_1) + \imath_{x_1}\imath_{x_2}H_{(2)})
\end{aligned}
$$

Generalization to principal torus bundles

We have

$$H = H_{(3)} + A_i \wedge H^i_{(2)} + \frac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \frac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)}$$

such that

$$\begin{aligned} d_H = & \bar{d} + H_{(3)} + F_{(2)i} \partial_{A_i} + \frac{1}{2} F_{(1)ij} \partial_{A_i} \wedge \partial_{A_j} + \frac{1}{6} F_{(0)ijk} \partial_{A_i} \wedge \partial_{A_j} \wedge \partial_{A_k} \\ & + A_i \wedge H^i_{(2)} + \frac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \frac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)} \end{aligned}$$

The $F_{(1)ij}$ and $F_{(0)ijk}$ are known as nongeometric fluxes

Nongeometric fluxes

Let $\{e_a\}$ be a basis of $\Gamma(TE)$, such that $[e_a, e_b] = f_{ab}{}^c e_c$, and $\{e^a\}$ be a dual basis of $\Gamma(T^*E)$, then the Courant bracket can be expressed as

$$\begin{aligned} [\![e_a, e_b]\!] &= f_{ab}{}^c e_c + h_{abc} e^c \\ [\![e_a, e^b]\!] &= q^{bc}{}_a e_c - f_{ac}{}^b e^c \\ [\![e^a, e^b]\!] &= 0 r^{abc} e_c + q^{ab}{}_c e^c \end{aligned}$$

where $H = \frac{1}{6} h_{abc} e^a \wedge e^b \wedge e^c$.
Together with certain conditions on the structure constants this defines a Courant algebroid.

Theorem [Bouwknegt-Garretson-Kao]: T-duality provides an isomorphism of (certain) Courant algebroids.

Nonlinear σ-model

Let $X : \Sigma \to M$ (Σ is 2D worldsheet, M is target space). Suppose (M, G) is a Riemannian manifold. Then a natural action is

$$
\begin{aligned}
S &= \tfrac{1}{2}\int_\Sigma G_{ij}(X)\, dX^i \wedge *dX^j \\
&= \tfrac{1}{2}\int d^2\sigma\; G_{ij}(X)(\dot{X}^i\dot{X}^j - X'^i X'^j)
\end{aligned}
$$

where $\dot{X}^i = \partial_\tau X^i$, $X'^i = \partial_\sigma X^i$.

The canonical momenta are given by

$$
P_i = \frac{\delta S}{\delta \dot{X}^i} = G_{ij}\dot{X}^j
$$

and the Hamiltonian is

$$
\begin{aligned}
H &= \int d\sigma\,(P_i\dot{X}^i - \mathcal{L}) \\
&= \tfrac{1}{2}\int d\sigma\,(G^{ij}P_iP_j + G_{ij}X'^i X'^j)
\end{aligned}
$$

Nonlinear σ-model

Poisson brackets

$$\{X^i(\sigma), X^j(\sigma')\}_{PB} = 0$$
$$\{P_i(\sigma), X^j(\sigma')\}_{PB} = \delta_i^j\, \delta(\sigma - \sigma')$$
$$\{P_i(\sigma), P_j(\sigma')\}_{PB} = 0$$

Related to a vector field $u = u^i \partial_i \in \Gamma(TM)$ we have currents

$$J_u = u^i P_i$$

and corresponding charges

$$Q_u = \int d\sigma\, J_u(\sigma)$$

generating transformations

$$\delta_u X^i = \{Q_u, X^i\}_{PB} = u^i$$

Nonlinear σ-model

The charge algebra is given by

$$\begin{aligned}\{Q_u, Q_v\}_{PB} &= \int d\sigma d\sigma' \{(u^i P_i)(\sigma), (v^i P_i)(\sigma')\} \\ &= \int d\sigma\, [u, v]^i P_i = Q_{[u,v]}\end{aligned}$$

We have

$$\{Q_u, H\}_{PB} = \int d\sigma\, \mathcal{L}_u(G_{ij}) \left(G^{ik} G^{jl} P_k P_l - X'^k X'^l \right)$$

hence the charge is conserved if $\mathcal{L}_u(G_{ij}) = 0$, i.e. if u is a Killing vector for the metric G.

Nonlinear σ-model

Now, for $(u,\xi) \in \Gamma(TM \oplus T^*M)$ consider the generalized current

$$J_{(u,\xi)} = u^i P_i + \xi_i X'^i$$

and charges

$$Q_{(u,\xi)} = \int d\sigma \, J_{(u,\xi)}$$

A similar computation as before gives

$$\{Q_{(u,\xi)}, Q_{(v,\eta)}\}_{PB} = Q_{[\![(u,\xi),(v,\eta)]\!]}$$

where

$$[\![(u,\xi),(v,\eta)]\!] = ([u,v], \mathcal{L}_u \eta - \mathcal{L}_v \xi - \tfrac{1}{2} d(\imath_u \eta - \imath_v \xi))$$

is the Courant bracket.

Nonlinear σ-model

while

$$\{Q_{(u,\xi)}, H\}_{PB} = \int d\sigma \left(\mathcal{L}_u(G_{ij})(G^{ik}G^{jl}P_kP_l - X'^k X'^l) \right.$$
$$\left. + 2\left(\partial_i\xi_j - \partial_j\xi_i\right) G^{ik}P_kX'^j \right)$$

hence the charge is conserved if $\mathcal{L}_u(G_{ij}) = 0$ and $d\xi = 0$.

Nonlinear σ-model with B-field

We can 'twist' the nonlinear σ-model by a so-called B-field, $B \in \Omega^2(M)$

$$S = \frac{1}{2} \int \left(G_{ij}(X)\, dX^i \wedge *dX^j + B_{ij}(X) dX^i \wedge dX^j \right)$$
$$= \int d^2\sigma \, \left(\tfrac{1}{2} G_{ij}(X)(\dot{X}^i \dot{X}^j - X'^i X'^j) - B_{ij} \dot{X}^i X'^j \right)$$

The canonical momenta are given by

$$P_i = \frac{\delta S}{\delta \dot{X}^i} = G_{ij} \dot{X}^j - B_{ij} X'^j$$

and the Hamiltonian is

$$H = \frac{1}{2} \int d\sigma \, (G^{ij}(P_i + B_{ik} X'^k)(P_j + B_{jl} X'^l) + G_{ij} X'^i X'^j)$$

Nonlinear σ-model with B-field

The charges

$$Q_{(u,\xi)} = \int d\sigma\, J_{(u,\xi)} = \int d\sigma\, \left(u^i (P_i + B_{ik} X'^k) + \xi_i X'^i \right)$$

now satisfy

$$\{Q_{(u,\xi)}, Q_{(v,\eta)}\}_{PB} = Q_{[\![(u,\xi),(v,\eta)]\!]_H}$$

where $H = dB$ and

$$[\![(u,\xi),(v,\eta)]\!]_H = ([u,v], \mathcal{L}_u \eta - \mathcal{L}_v \xi - \tfrac{1}{2} d(\imath_u \eta - \imath_v \xi) + \imath_u \imath_v H)$$

is the twisted Courant bracket.

while

$$\{Q_{(u,\xi)}, H\}_{PB} = \int d\sigma \Big(\mathcal{L}_u(G_{ij})(G^{ik}G^{jl}P_kP_l - X'^iX'^j) + 2\Big(-u^kH_{ijk} + (\partial_i\xi_j - \partial_j\xi_i)\Big)G^{ik}P_kX'^j\Big)$$

hence the charge is conserved if $\mathcal{L}_u(G_{ij}) = 0$ and $\imath_u H = d\xi$ (implying $\mathcal{L}_u H = 0$ and $\mathcal{L}_u B = d(\xi - \imath_u B)$).

Remark: Model can be generalized to SUSY σ-models (related to complex structures on M).

Poisson σ-model

Let (M, Π) be a Poisson manifold. I.e. $\Pi = \frac{1}{2}\Pi^{ij}\partial_i \wedge \partial_j$ is a Poisson bivector

$$[\Pi, \Pi]_S^{ijk} = \Pi^{il}\partial_l \Pi^{jk} + \text{cycl} = 0$$

Let $A \in \Gamma(T^*\Sigma \otimes X^*(T^*M))$. I.e. $A(\sigma)$ is a 1-form on Σ with values in $T^*_{X(\sigma)}M$.
The Poisson σ-model is defined by

$$S = \int_\Sigma \left(A_i \wedge dX^i + \tfrac{1}{2}\Pi^{ij} A_i \wedge A_j \right)$$

The equations of motion are

$$\begin{aligned} dX^i + \Pi^{ij} A_j &= 0 \\ dA_i + \tfrac{1}{2}\partial_i \Pi^{kl} A_k \wedge A_l &= 0 \end{aligned}$$

[The Poisson condition arises as a consistency equation for the equations of motion]

Poisson σ-model

We can write the action of the Poisson σ-model as

$$S = \int d^2\sigma \left(P_i \dot{X}^i - A_{i\tau}\phi^i \right)$$

where

$$P_i = A_{i\sigma}\,, \qquad \phi^i = X'^i + \Pi^{ij} P_j$$

This shows that the Poisson σ-model is a constrained dynamical system with vanishing Hamiltonian, and that $P_i = A_{i\sigma}$ is the momentum conjugate to X^i.
We have

$$\{\phi^i(\sigma), \phi^j(\sigma')\}_{PB} = -(\partial_k \Pi^{ij})\phi^k(\sigma)\delta(\sigma - \sigma')$$

(provided $[\Pi, \Pi]_S = 0$). I.e. the constraints are first class.

Poisson σ-model

Symmetries parametrized by $\xi = \xi_i dX^i \in \Gamma(X^*(T^*M))$

$$\delta_\xi X^i = -\Pi^{ij}\xi_j$$
$$\delta_\xi A_i = d\xi_i + \partial_i \Pi^{jk} A_j \xi_k$$

The corresponding current is given by

$$J_\xi = \xi_i (X'^i + \Pi^{ij} P_j)$$

and, after a short calculation

$$\{Q_\xi, Q_\eta\} = Q_{[\xi,\eta]_\Pi}$$

where

$$[\xi,\eta]_\Pi = \mathcal{L}_{\Pi\xi}\eta - \mathcal{L}_{\Pi\eta}\xi + d(\Pi(\xi,\eta))$$

is the Koszul bracket on $\Gamma(T^*M)$ and we have identified the bi-vector Π with $\Pi : T^*M \to TM$.

Poisson σ-model (cont'd)

We can interpret the fields (A, X) of the Poisson σ-model as a defining a bundle map

$$\begin{array}{ccc} T\Sigma & \xrightarrow{A} & T^*M \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{X} & M \end{array}$$

Theorem: The bundle map $\Phi = (A, X)$ is a Lie algebroid morphism iff Φ is a solution to the Poisson σ-model equations of motion.

Generalized WZW Poisson σ-model

Consider

$$S = \int_\Sigma \Big(A_i \wedge dX^i + \tfrac{1}{2}\big(G^{ij} A_i \wedge * A_j + \Pi^{ij} A_i \wedge A_j + B_{ij} dX^i \wedge dX^j\big)\Big)$$

The resulting Hamiltonian is

$$\begin{aligned} H =& \tfrac{1}{2} \int d\sigma \,\Big(G^{ij}(P_i + B_{ik} X'^k)(P_j + B_{jl} X'^l) \\ & + G_{ij}(X'^i + \Pi^{ik}(P_k + B_{kr} X'^r))(X'^j + \Pi^{jl}(P_l + B_{ls} X'^s))\Big) \end{aligned}$$

where $P_i = A_{i\sigma}$.

Generalized Poisson σ-model

The currents/charges

$$J_{(u,\xi)} = u^i(P_i + B_{ik}X'^k) + \xi_i(X'^i + \Pi^{ik}(P_k + B_{kl}X'^l))$$

satisfy

$$\{Q_{(u,\xi)}, Q_{(v,\eta)}\}_{PB} = Q_{[\![(u,\xi),(v,\eta)]\!]_R}$$

where, in terms of a (local) basis $e_i = \partial_i$, $e^i = dX^i$ of $\Gamma(TM \oplus T^*M)$

$$\begin{aligned}
[\![e_i, e_j]\!]_R &= f_{ij}{}^k e_k + h_{ijk} e^k \\
[\![e_i, e^j]\!]_R &= q_i{}^{jk} e_k - f_{ik}{}^j e^k \\
[\![e^i, e^j]\!]_R &= r^{ijk} e^k + q_k{}^{ij} e^k
\end{aligned}$$

Generalized Poisson σ-model

with, the so-called Roytenberg relations

$$
\begin{aligned}
h_{ijk} &= \partial_{[i} B_{jk]} \\
f_{ij}{}^{k} &= -h_{ijl} \Pi^{lk} \\
q_k{}^{ij} &= \partial_k \Pi^{ij} - h_{klm} \Pi^{li} \Pi^{mj} \\
r^{ijk} &= [\Pi, \Pi]_S^{ijk} + h_{lmn} \Pi^{li} \Pi^{mj} \Pi^{nk}
\end{aligned}
$$

This bracket is a particular example of a Courant algebroid (a so-called Π-twisted Courant algebra).

WZ term

It is of course well-known that the B-term can be lifted to three dimensions, i.e. only needs to be locally defined, provided $H = dB$ is a globally defined form, then

$$\frac{1}{6}\int_N H_{ijk}\, dX^i \wedge dX^j \wedge dX^k = \frac{1}{2}\int_{\Sigma=\partial N} B_{ij}\, dX^i \wedge dX^j$$

(cf. WZW model).

Generalized WZ terms

More generally, one can lift the other topological terms as well and consider a generalized WZW Poisson σ-model

$$
\begin{aligned}
S = \int_{\Sigma=\partial N} & \left(A_i \wedge d\tilde{X}^i + \tfrac{1}{2} G^{ij} A_i \wedge * A_j \right) \\
& + \int_N \Big(\tfrac{1}{6} h_{ijk} d\tilde{X}^i \wedge d\tilde{X}^j \wedge d\tilde{X}^k + \tfrac{1}{2} f_{ij}{}^k A_k \wedge d\tilde{X}^i \wedge d\tilde{X}^j \\
& + \tfrac{1}{2} q_k{}^{ij} A_i \wedge A_j \wedge d\tilde{X}^k + \tfrac{1}{6} r^{ijk} A_i \wedge A_j \wedge A_k \Big)
\end{aligned}
$$

where $d\tilde{X}^i = dX^i + \Pi^{ij} A_j$.

It turns out that this generalized WZ-term is topological (locally a total derivative) iff the Roytenberg relations are satisfied.

HAPPY BIRTHDAY ALAN!