EFFECTIVE RATNER THEOREM FOR $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ AND GAPS IN $\sqrt{n}$ MODULO 1

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Abstract. Let $G = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Building on recent work of Strömbergsson we prove a rate of equidistribution for the orbits of a certain 1-dimensional unipotent flow of $\Gamma\backslash G$, which projects to a closed horocycle in the unit tangent bundle to the modular surface. We use this to answer a question of Elkies and McMullen by making effective the convergence of the gap distribution of $\sqrt{n}$ mod 1.

Results of Ratner on measure rigidity and equidistribution of orbits [4, 5] play a fundamental role in the study of unipotent flows on homogeneous spaces. They have many applications beyond the world of dynamics, ranging from problems in number theory to mathematical physics. This paper is concerned with the problem of obtaining *effective* versions of results that build on Ratner's theorem and is inspired by recent work of Strömbergsson [8].

Let $G = \mathrm{ASL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ be the group of affine linear transformations of $\mathbb{R}^2$. We define the product on G by

$$(M,\boldsymbol{x})(M',\boldsymbol{x}') = (MM', \boldsymbol{x}M' + \boldsymbol{x}'),$$

and the right action is given by $\boldsymbol{x}(M,\boldsymbol{x}') = \boldsymbol{x}M + \boldsymbol{x}'$. We always think of $\boldsymbol{x} \in \mathbb{R}^2$ as a row vector. Put $\Gamma = \mathrm{ASL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ and let $X = \Gamma\backslash G$ be the associated homogeneous space. The group G is unimodular and so the Haar measure μ on G projects to a right-invariant measure on X. The space X is non-compact, but it has finite volume with respect to the projection of μ. Following the usual abuse of notation, we denote the projected measure by μ and normalize it so that $\mu(X) = 1$.

Let

$$a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix},$$

and write $A^+ = \{a(y) \colon y > 0\}$. In what follows we will use the embedding $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow G$, given by $M \mapsto (M, \mathbf{0})$, which thereby allows us to think of $\mathrm{SL}(2,\mathbb{R})$ as a subgroup of G. Strömbergsson [8] works with the unipotent flow on X generated by right multiplication by the subgroup

$$U_0 = \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, (0,0) \right) : x \in \mathbb{R} \right\}.$$

He considers orbits of a point $(\mathrm{Id}_2, (\xi_1, \xi_2))$ subject to a certain Diophantine condition. In [8, Thm. 1.2], effective rates of convergence are obtained for the equidistribution of such orbits under the flow $a(y)$ as $y \to 0$. The goal of the present paper is to extend the methods of Strömbergsson to handle the orbit generated by right multiplication by the subgroup $U = \{u(x) : x \in \mathbb{R}\}$, where

$$u(x) = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \left(\frac{x}{2}, \frac{x^2}{4} \right) \right).$$

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As noted by Strömbergsson [8, §1.3], *any* Ad-unipotent 1-parameter subgroup of G with non-trivial image in $\mathrm{SL}(2,\mathbb{R})$ is conjugate to either U_0 or U.

With this notation we will study the rate of equidistribution of the closed orbit $\Gamma\backslash\Gamma U$ under the action of $a(y)$, as $y \to 0$. Geometrically this orbit is a lift of a closed horocycle in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ to $\Gamma\backslash G$, and the behaviour of horocycles under the flow A^+ on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ is very well understood. The main obstruction to treating the problem of horocycle lifts with the usual techniques of ergodic theory (such as thickening) is the fact that U is *not* the entire unstable manifold of the flow $a(y)$, but only a codimension 1 submanifold. Elkies and McMullen [3] used Ratner's measure classification theorem [4] to prove that the horocycle lifts equidistribute, but their method is ineffective. In [3, §3.6] they ask whether explicit error estimates can be obtained. The following result answers this affirmatively.

Theorem 1. *There exists $C > 0$ such that for every $f \in C_{\mathrm{b}}^8(X)$ and $y > 0$ we have*

$$\left|\frac{1}{2}\int_{-1}^{1} f(u(x)a(y))\,dx - \int_X f\,d\mu\right| < C\|f\|_{C_{\mathrm{b}}^8} y^{\frac{1}{4}} \log^2(2+y^{-1}).$$

Here $C_{\mathrm{b}}^k(X)$ denotes the space of k times continuously differentiable functions on X whose left-invariant derivatives up to order k are bounded.

Our next result shows that we can replace dx by a sufficiently smooth absolutely continuous measure. Let $\rho\colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ be a compactly supported function that has $1+\varepsilon$ derivatives in L^1. For simplicity we follow [8] and interpolate between the Sobolev norms $\|\rho\|_{W^{1,1}}$ and $\|\rho\|_{W^{2,1}}$, which give the L^1 norms of first and second derivatives, respectively. This interpolation allows us to treat the case of piecewise constant functions with an ε-loss in the rate.

Theorem 2. *Let $\eta \in (0,1)$. There exists $K > 1$ and $C(\eta) > 0$ such that for every $f \in C_{\mathrm{b}}^8(X)$ and $y > 0$ we have*

$$\left|\int_{\mathbb{R}} f(u(x)a(y))\rho(x)dx - \int_X f\,d\mu \int_{\mathbb{R}} \rho(x)\,dx\right| < C(\eta)\|\rho\|_{W^{1,1}}^{1-\eta}\|\rho\|_{W^{2,1}}^{\eta}\|f\|_{C_{\mathrm{b}}^8} y^{\frac{1}{4}} \log^{K-1}(2+y^{-1}).$$

The constant K in this result is absolute and does not depend on η. The proof of Theorems 1 and 2 builds on the proof of [8, Thm. 1.2]. It relies on Fourier analysis and estimates for complete exponential sums which are essentially due to Weil. Let us remark that while we strive to obtain the best possible decay in y, we take little effort to optimize the norms of f and ρ that appear in the estimates. The exponent $\frac{1}{4}$ in the error term is optimal for our method, but we surmise it can be improved by exploiting additional cancellation in certain two dimensional exponential sums. The natural upper limit is $\frac{1}{2}$, which holds for horocycles on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ due to work of Sarnak [6].

We may apply Theorem 1 to study gaps between the fractional parts of $\sqrt{n}$. Consider the sequence $\sqrt{n} \bmod 1 \subset \mathbb{R}/\mathbb{Z} \cong S^1$. It is easy to see from Weyl's criterion that this sequence is uniformly distributed on the circle. This means that for every interval $J \subset S^1$, we have

$$\lim_{N\to\infty} \frac{\#\{\sqrt{n} \bmod 1\colon 1 \leqslant n \leqslant N\} \cap J}{N} = |J|,$$

where $|\cdot|$ denotes length. The statistic we focus on is the *gap distribution.* For each $N \in \mathbb{N}$, we consider the set $\{\sqrt{n} \bmod 1\}_{1\leqslant n\leqslant N}$ and we allow $0 \in \mathbb{R}/\mathbb{Z}$ to be included for each perfect square. This set of N points divides the circle into N intervals (a few of which could be of zero length) which we refer to as *gaps*. For $t \geqslant 0$, we define the *gap distribution* $\lambda_N(t)$ to be the proportion of gaps whose length is less than t/N. This function satisfies $\lambda_N(0) = 0$ and $\lambda_N(\infty) = 1$, and it is left-continuous.

The behaviour of $\lambda_N(t)$, as $N \to \infty$, has been analyzed by Elkies and McMullen [3] and later also by Sinai [7]. It is shown in [3] that there exists a function $\lambda_\infty(t)$ such that $\lambda_N(t) \to \lambda_\infty(t)$ for each t. We have

$$\lambda_\infty(t) = \int_0^t F(\xi)\, d\xi, \tag{0.1}$$

where F is given in [3, Thm. 1.1]. It is defined by analytic functions on three intervals, but it is not analytic at the endpoints joining these intervals. Moreover, it is constant on the interval $[0, 1/2]$.

The approach of Elkies and McMullen [3] is to relate $\lambda_N(t)$ to a function on X, so that the problem of understanding $\lambda_N(t)$ is translated into studying

$$\frac{1}{2}\int_{-1}^{1} f(u(x)a(1/N))\, dx,$$

as $N \to \infty$, for a certain function f that depends on t. The error terms appearing in this step are worked out explicitly in [3]. In fact, f is directly related to

$$\sigma_N(t) = \int_0^t \xi d\lambda_N(\xi), \tag{0.2}$$

which is the total length of gaps whose length is less than t/N. The key input in [3] comes from Ratner's theorem [4], which is used in [3, Thm. 2.2] to find the limiting distribution of $\sigma_N(t)$. Armed with Theorem 1, we will refine this approach to get the following result.

Corollary 3. *Let $\lambda_N(t), \lambda_\infty(t)$ be as above and let $t \geqslant N^{-1/40}\log N$. Then there exists a function $C(t)$ such that*

$$|\lambda_N(t) - \lambda_\infty(t)| \ll C(t) N^{-1/36} \log^{2/9} N$$

for any $N \geqslant 2$.

The sequence $\sqrt{n} \bmod 1$ has also been studied from the perspective of its *pair correlation function*. This is a useful statistic for measuring randomness in sequences and, in this setting, it has been shown to converge to that of a Poisson point process by El-Baz, Marklof, and the second author [2]. In the light of Theorem 1, although we will not carry out the details here, by developing effective versions of the results in [2] it would be possible to conclude that the pair correlation function converges *effectively*. By way of comparison, we remark that Strömbergsson [8, §1.3] indicates how one might make effective the convergence of the pair correlation function in the problem of *visible lattice points* (see [1]).

References

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