JARNÍK-TYPE INEQUALITIES FOR BOUNDED ORBITS

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Let $X = (X, d)$ be a metric space and $T : X \to X$ a continuous transformation. Let $\mathcal{O} \in X$ be an *obstacle*.[1] For a subset $S \subset X$ we obtain the quadruple $\mathcal{D} = (X, T, \mathcal{O}, S)$ and consider the set

$$\textbf{Bad}_{\mathcal{D}} \equiv \{x \in S : c(x) = \inf_{n \in \mathbb{N}_0} d(T^n(x), \mathcal{O}) > 0\}$$

of points in S for which the orbit avoids some (open) ball around $\mathcal{O}$ of radius c depending on x. Notice that when μ is an ergodic Borel measure with respect to T and $\mathcal{O}$ lies in the support of μ then $\textbf{Bad}_{\mathcal{D}}$ is a μ-null set. The set $\textbf{Bad}_{\mathcal{D}}$ has been studied for several examples with different techniques, for instance via Schmidt's game and its winning sets (see for instance [3, 13]), choosing 'nice fractal sets' S, or in terms of properties of the *(Markoff) spectrum* $\mathcal{S}_{\mathcal{D}} \equiv \{c(x) : x \in S\}$ (see [2, 7, 10]). As a result various qualitative properties have been achieved, such as full Hausdorff dimension, a property of winning sets in a reasonably nice setting. For a given small $c > 0$, our goal is to determine *quantitative* results on the dimension of the set

$$\textbf{Bad}_{\mathcal{D}}(c) \equiv \{x \in S : T^n(x) \notin B(\mathcal{O}, c) \text{ for all } n \in \mathbb{N}_0\}.$$

Recalling the well-known correspondence between bounded geodesic rays on the modular surface $\mathbb{H}^2/SL(2, \mathbb{Z})$ and badly approximable numbers in $\mathbb{R}$, a previous result on this question is due to [4]. In fact, for $n \geq 1$ denote the set of *badly approximable vectors* (or badly approximable numbers for $n = 1$) by

$$\textbf{Bad}_{\mathbb{R}^n} \equiv \{\bar{x} \in \mathbb{R}^n : c(\bar{x}) \equiv \inf_{(\bar{p},q) \in \mathbb{Z}^n \times \mathbb{N}} q^{1/n} \|q\bar{x} - \bar{p}\| > 0\},$$

and set $\textbf{Bad}_{\mathbb{R}^n}(c) \equiv \{\bar{x} \in \textbf{Bad}_{\mathbb{R}^n} : c(\bar{x}) \geq c^{1/n}\}$. In [14] we prove the following.

Theorem 0.1. *There exist positive constants k_l, k_u and $c_0 > 0$, depending only on n, such that for all $0 < c < c_0$ we have*

$$n - k_l \cdot \frac{c^{1/(2n)}}{|\log(c)|} \leq \dim(\textbf{Bad}_{\mathbb{R}^n}(c)) \leq n - k_u \cdot \frac{c^{n+1}}{|\log(c)|}.$$

For $n = 1$, a similar inequality was shown by Jarník [4], which we call *Jarník's inequality*. Recently, [1] extended the result to the set of badly approximable matrices and also improved the upper bound. However, our proof of Theorem 0.1 relies on an axiomatic approach in a general setup with a Diophantine setting, as in [5]. Given a family of 'resonant sets' together with a height and approximation function, we define a set of badly approximable elements in an abstract fashion. The required conditions, stated in terms of the resonant sets, are deduced from 'local measure and intersection conditions' and are satisfied for various examples.

Let us present a further applications of this axiomatic approach. Let M be a complete $(n+1)$-dimensional finite volume hyperbolic manifold with precisely one cusp (that we

[1] Instead of a point, we may alternatively consider an object, such as a topological end of X or a set in X, providing suitable neighborhoods in X (see the examples below).

view as the obstacle). For a point $o \in M$ let SM_o be the n-dimensional unit tangent space of M at o. Identify a vector $v \in SM_o$ with the unique geodesic $\gamma_v : \mathbb{R}_{>0} \to M$ starting at o such that $\dot\gamma_v(0) = v$. Associated to the cusp, consider a (Busemann) height function $\beta : M \to \mathbb{R}$ which defines the cusp neighborhoods $H_t := \beta^{-1}((t,\infty))$ at height t. For $t \geq 0$ define the set of rays γ_v, $v \in SM_o$, which avoid the cusp neighborhood H_t by

$$\begin{aligned}\mathbf{Bad}_{M,H_0,o}(t) &\equiv \{v \in SM_o : \gamma_v(s) \notin H_t \text{ for all } s \geq 0\}, \\ &= \{v \in SM_o : \mathcal{H}(v) := \sup_{s\geq 0} \beta(\gamma_v(s)) \leq t\}.\end{aligned}$$

Theorem 0.2 ([15]). *There exist positive constants k_l, k_u and a height $t_0 \geq 0$, depending on M and the choices of H_0 and o, such that for all $t > t_0$ we have*

$$n - \frac{k_l}{t \cdot e^{n/2t}} \leq \dim(\mathbf{Bad}_{M,H_0,o}(t)) \leq n - \frac{k_u}{t \cdot e^{2nt}}.$$

A similar inequality holds for M geometrically finite. The set of 'bounded' rays follows to be of full Hausdorff dimension, earlier shown by [11], and is even an absolute winning set, see [8]. The height spectrum $\mathcal{S} := \{\mathcal{H}(v) : v \in SM_o\}$ was shown to contain a *Hall ray* (see [12] for $n = 1$ and [9] for $n \geq 2$), that is, there is an interval $(t_0, \infty) \subset \mathcal{S}$. Define the function $\mathfrak{D}$ on $\mathcal{S}$ by

$$\mathfrak{D} : \mathcal{S} \to [0, n], \quad t \mapsto \dim(\{v \in SM_o : \mathcal{H}(v) = t\}).$$

Clearly, if $\mathfrak{D}(t) > 0$ then $t \in \mathcal{S}$. In [16] we give a lower bound for $\mathfrak{D}$ for $n \geq 2$.

Theorem 0.3 ([16]). *There exists a height $t_0 \geq 0$ such that $[t_0, \infty) \subset \mathcal{S}$ and a positive constant k_0 such that for $t \in [t_0, \infty)$ we have*

$$\mathfrak{D}(t) \geq (n-1) - \frac{k_0}{t}.$$

Using the axiomatic approach further Jarník-type inequalities (and generalizations of the above ones) are achieved for

1. the set $\mathbf{Bad}_{\mathbb{R}^n}(\bar r)$ of badly approximable vectors with weight vector $\bar r$, [15];
2. the set of geodesic rays in a complete (finite volume or geometrically finite) hyperbolic manifold for which the obstacle is
   a) a point, [16], or
   b) a closed geodesic (or suitable higher dimensional totally geodesic submanifold), [15];
3. the set of Teichmüller geodesics staying in compact subsets of the stratum of translation surfaces, [6],
4. the set of orbits of a toral endomorphism which avoid a point in T^n, [15], and
5. the set of words in the Bernoulli shift which avoid a periodic word, [15].

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