"Mein Vater, mein Vater, und hoerest du nicht,
Was Erlenkoenig mir leise verspricht?"
"Sei ruhig, bleib ruhig, mein Kind!
In duerren Blaettern saeuselt der Wind."
J. W. Goethe

CRIES AND WHISPERS IN WINDTREE FORESTS

VINCENT DELECROIX AND ANTON ZORICH

Abstract. We study billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of a right-angled polygonal shape. One of our mane interests is dependence of the diffusion rate of the billiard on the number of components and on the shape of the obstacle. We prove, in particular, that when the number of angles of a symmetric connected obstacle grows, the diffusion rate tends to zero.

Our results are based on computation of Lyapunov exponents of the Hodge bundle over hyperelliptic loci in the moduli spaces of quadratic differentials, which represents independent interest. In particular, we study hyperelliptic loci over the stratum with only simple zeros and poles in genus zero and compute explicit asymptotics for the sum of the Lyapunov exponents in two opposite regimes: when most of simple poles are ramified and when only few simple poles are ramified.

The computation of Lyapunov exponents uses, in particular, certain new combinatorial identities for hypergeometric sums.

1. Introduction

The wind-tree corresponds to a billiard in the plane endowed with a $\mathbb{Z}^2$-periodic obstacles of rectangular shape; the sides of the rectangles are aligned along the lattice, see Figure 1.

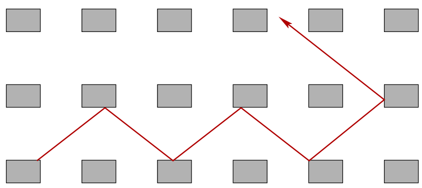

Figure 1. Original windtree model.

The wind-tree model was introduced by P. Ehrenfest and T. Ehrenfest [Eh] and studied by J. Hardy and J. Weber [HaWe] who had physical motivations.

Several advances were obtained recently using the powerful technology of deviation spectrum of measured foliations on surfaces and the underlying dynamics in the moduli space. For all parameters of the obstacle and for almost all directions the trajectories are known to be nonclosed and recurrent [AH]; there are examples of divergent trajectories constructed in [D]; the non-ergodicity is proved in [FU]. It was proved in [DHL] that the diffusion rate is $\frac{2}{3}$, and does not depend neither on the concrete values parameters of the obstacle nor on almost any direction and starting point , see Figure 2.

In other words, the maximal deviation of the trajectory from the starting point during the time t has the order of $t^{\frac{2}{3}}$ for large t. Thus, this behavior is quite

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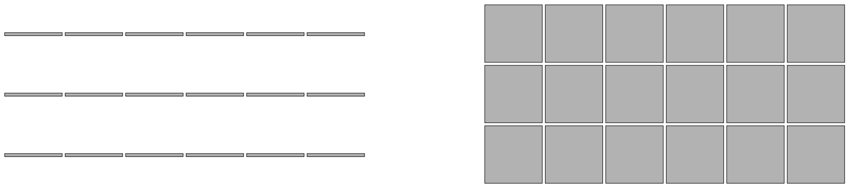

FIGURE 2. The diffusion rate $\frac{2}{3}$ does not depend on particular values of the parameters of the rectangular scatterer: it is the same both for narrow periodic corridors and for the plane with horizontal walls having tiny periodic holes.

different from the brownian motion, random walk in the plane, or billiards in the plane with periodic dispersing scatterers: for all of them the diffusion has the order $\sqrt{t}$ (and, thus, the diffusion rate is $\frac{1}{2}$).

We address the natural question "what happens if we change the shape of the obstacle?". We do not have ambition to solve this problem in the most general setting. We consider several interesting families of obstacles which represent different behavior and study the properties of the corresponding windtrees as the combinatorics of the obstacle inside the family becomes more complicated. We show, in particular, that if the obstacle is a connected right-angled polygon as on Figure 3, then the diffusion rate in the corresponding windtree model tends to zero as the number of corners of the obstacle grows; see Theorem 1 for more precise statement. This result gives an explicit affirmative answer to a question addressed by J.-C. Yoccoz.

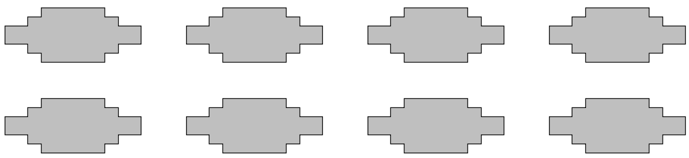

FIGURE 3. The diffusion rate in this windtree forest tends to zero when the number of corners of the obstacle grows.

For some families of obstacles (in particular, for the family as in Figure 3) we also count periodic trajectories and the generalized diagonals, and find explicit quadratic asymptotics for these counting problems.

We apply the technique from dynamics in the moduli space to resolve the original billiard problem. To describe families of windtrees we study the associated families of hyperelliptic loci in the moduli spaces of quadratic differentials. Using recent results from [AEZ] where all Siegel–Veech constants were explicitly computed for the moduli spaces of quadratic differentials in genus zero, we compute the Siegel–Veech constants for the hyperelliptic loci. Part of the computation relies on certain combinatorial identities. Together this allows us to compute the sum Λ^+ of the Lyapunov exponents of the Hodge bundle H^1_+ over the corresponding hyperelliptic loci with respect to the Teichmüller geodesic flow.

The current technology of Teichmüller dynamics is applicable to the moduli spaces of holomorphic 1-forms and to the moduli spaces of meromorphic quadratic differentials with at most simple poles. The natural parameter of complexity of a stratum of holomorphic 1-forms is the genus of the underlying Riemann surface while for the moduli spaces of quadratic differentials the number of simple poles serves as an extra parameter. Our results provide certain evidence that when the genus is fixed and the number of simple poles grows, the Lyapunov exponents of the Hodge bundle tend to zero (see [GrHu] in this connection). We explicitly compute the asymptotics for the sum Λ_+ of positive Lyapunov exponents for the hyperelliptic locus over the stratum $\mathcal{Q}(1^m, -1^{m+4})$ when the cover has small number of ramification points.

The transition from billiard dynamics to dynamics in the moduli space is described in the original paper [DHL] for the aspects concerning the generic trajectories, and in [AEZ] in what concerns the closed trajectories. In the current paper we cover only complementary issues. One of additional difficulties is related to the fact that for many windtree families , the dimension of subvariety $\mathcal{B}$ of flat surfaces arising from billiards is smaller than half dimension of the ambient invariant locus $\mathcal{Q}$ in the moduli space of quadratic differentials. In this situation we provide extra arguments proving that nevertheless the $\mathrm{PSL}(2,\mathbb{R})$-orbit closure of $\mathcal{B}$ is still entire $\mathcal{Q}$. Combining the standard transversality arguments and recent results [CE] we then prove that almost all flat surfaces in $\mathcal{B}$ share the same Lyapunov spectrum as almost all flat surfaces in $\mathcal{Q}$.

As often in the problems related to Siegel–Veech constants of the invariant loci in the moduli spaces of quadratic differentials (compare to [AEZ]) at some stage we run into certain combinatorial problems. Namely, we have to find the values of the sums of the kind

$$\sum_{k=0}^{m} \frac{\binom{m+p_1}{k+q_1}\binom{m+p_2}{k+q_2}}{\binom{2(m+p_3)}{2(k+q_3)}}.$$

or

$$\sum_{k=0}^{m} \frac{\binom{2(m+p_1)}{2(k+q_1)}\binom{2(m+p_2)}{2(k+q_2)}}{\binom{4(m+p_3)}{4(k+q_3)}},$$

where p_i, q_i are integer parameters. We find the combinatorial identities for the values of relevant sums. In most cases some elementary manipulations allow us to transform the sum to an expression $S(m)$ for which the Zeilberger's creative telescoping algorithm [Ze1], [Ze2] provides a simple relation of the form

$$S(m+1) = Q(m)S(m)\,,$$

where $Q(m)$ is already an explicit rational function. The same algorithm provides also the certificate function $R(m,k)$ which proves the relation above. In certain cases we use computer assistant implementation [PSR] of Zeilberger's algorithm.

2. Main results

Denote by $\mathcal{B}(m)$ the family of billiards such that the obstacle has $4m$ corners with the angle $\pi/2$. Say, all billiards from the original windtree family as in Figures 1 and 2 live in $\mathcal{B}(1)$; the billiard in Figure 3 belongs to $\mathcal{B}(3)$; the billiard in Figure 3 belongs to $\mathcal{B}(17)$.

Theorem 1. *For almost all billiard tables in the family $\mathcal{B}(m)$ and for almost all directions the diffusion rate $\delta(m)$ is the same and equals*

$$\delta(m) = \frac{(2m)!!}{(2m+1)!!}\,.$$

When $m \to +\infty$ $\delta(m)$ has asymptotics

$$\delta(m) = \frac{\sqrt{\pi}}{2\sqrt{m}}\left(1 + O\left(\frac{1}{m}\right)\right)\,.$$

Here the double factorial means the product of all even (correspondingly odd) natural numbers from 2 to $2m$ (correspondingly from 1 to $2m+1$). For the original windtree, when the obstacle is a rectangle $m = 1$ we see the value $\delta(1) = \frac{2}{3}$ found in [DHL].

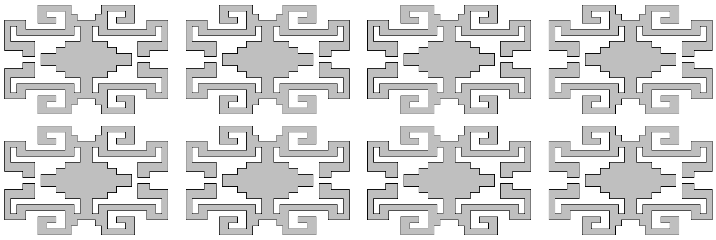

FIGURE 4. The diffusion rate depends only on the number of corners of the obstacle and not on the particular values of (almost all) length parameters nor on the particular shape of the obstacle.

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