A GENERALIZATION OF THE WIENER-WINTNER THEOREM

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The classical Wiener-Wintner theorem [10] states that for every measure-preserving system (X, μ, T) and every integrable function f on X there exists a full measure set $X' \subset X$ such that the weighted ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} \lambda^n f(T^n x)$$

converge for every $\lambda \in \mathbb{T}$, $\mathbb{T}$ the unit circle, and every $x \in X'$. By an observation of Bourgain, the above averages converge to zero uniformly in λ whenever f is orthogonal to the Kronecker factor. Moreover, for uniquely ergodic systems and continuous functions the convergence is also uniform in x, see Assani [1].

There are several proofs and various generalizations and extensions of this result. For example, a result of Lesigne [9] implies that the family of weights (λ^n), $\lambda \in \mathbb{T}$, can be replaced by $(\lambda_1^{p_1(n)} \ldots \lambda_d^{p_d(n)})$, $d \in \mathbf{N}$, $\lambda_1, \ldots \lambda_d \in \mathbb{T}$, $p_1, \ldots p_d \in Z[X]$. Frantzikinakis showed uniform convergence to zero of the corresponding weighted averages for f orthogonal to the Abramov factor for totally ergodic systems. Finally, Host and Kra generalized the Wiener-Wintner theorem to the class of nilsequences.

A sequence $(a_n) \subset \mathbb{C}$ is called a *(basic) nilsequence* if there exists a nilpotent Lie group G, a discrete cocompact subgroup Γ of G, $y \in G/\Gamma$, a rotation S on G/Γ and $g \in C(G/\Gamma)$ such that $a_n = g(S^n y)$ holds for every n. Such a system $(G/\Gamma, \text{Haar}, S)$ is called a *nilsystem*. Nilsystems and nilsequences play an important role in multiple ergodic theory and additive number theory, see Host, Kra [7], Green, Tao [4, 5] and Green, Tao, Ziegler [6].

The Wiener-Wintner type result of Host and Kra mentioned above left uniform convergence open. The main result of our talk, based on a joint work with Pavel Zorin-Kranich [2], is a quantitative estimate of the Wiener-Wintner averages

$$\frac{1}{N}\sum_{n=1}^{N} a_n f(T^n x)$$

for nilsequences (a_n) on a fixed nilmanifold G/Γ by the corresponding Gowers-Host-Kra seminorm of the function f uniform in S, y and g from a certain Sobolev class. As a consequence, we obtain uniform convergence to zero of the above averages for functions orthogonal to the corresponding Host-Kra factor. The estimate and the convergence are moreover uniform in x for uniquely ergodic systems and continuous functions f.

References

[1] I. Assani, *Wiener Wintner Ergodic Theorems.* World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[2] T. Eisner, P. Zorin-Kranich, *Uniformity in the Wiener-Wintner theorem for nilsequences*, Discrete Contin. Dyn. Syst. **33** (2013), 3497–3516.

[3] N. Frantzikinakis, *Uniformity in the polynomial Wiener-Wintner theorem*, Ergodic Theory Dynam. Systems **26** (2006), 1061-1071.

[4] B. Green, T. Tao, *Linear equations in primes*, Ann. of Math. **171** (2010), 1753–1850.

[5] B. Green and T. Tao, *The Möbius function is strongly orthogonal to nilsequences*, Ann. of Math. **175** (2012), 541–566.

[6] B. Green, T. Tao and T. Ziegler, *An inverse theorem for the Gowers* $U^{s+1}[N]$*norm*, Ann. of Math. **176** (2012), 1231–1372.

[7] B. Host, B. Kra, *Nonconventional ergodic averages and nilmanifolds*, Ann. of Math. **161** (2005), 397–488.

[8] B. Host, B. Kra, *Uniformity seminorms on* l^∞ *and applications*, J. Anal. Math. **108** (2009), 219–276.

[9] E. Lesigne, *Spectre quasi-discret et théorème ergodique de Wiener-Wintner pour les polynômes*, Ergodic Theory Dynam. Systems **13** (1993), 767–784.

[10] N. Wiener, A. Wintner, *Harmonic analysis and ergodic theory,* Amer. J. Math. **63** (1941), 415–426.