INFINITE DETERMINANTAL MEASURES

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ABSTRACT. Infinite determinantal measures introduced in this note are inductive limits of determinantal measures on an exhausting family of subsets of the phase space. Alternatively, an infinite determinantal measure can be described as a product of a determinantal point process and a convergent, but not integrable, multiplicative functional.

Theorem 4.1, the main result announced in this note, gives an explicit description for the ergodic decomposition of infinite Pickrell measures on the spaces of infinite complex matrices in terms of infinite determinantal measures obtained by finite-rank perturbations of Bessel point processes.

1. INTRODUCTION

1.1. **Outline of the main results.** In this section, our aim is to construct sigma-finite analogues of determinantal measures on spaces of configurations. In Theorem 4.1 of Section 4, infinite determinantal measures will be seen to arise in the ergodic decomposition of infinite unitarily-invariant measures on spaces of infinite complex matrices.

Informally, a configuration on the phase space E is an unordered collection of points (called *particles*) of E, possibly with multiplicities; the main assumption is that a bounded subset of E contain only finitely many particles of a given configuration.

To a function g on E assign its *multiplicative functional* Ψ_g on the space of configurations: the functional Ψ_g is obtained by multiplying the values of g over all particles of a configuration (see (5)). A probability measure on the space of configurations on E is uniquely characterized by prescribing the expectations of multiplicative functionals; for *determinantal* probability measures these expectations are given by special Fredholm determinants, see e.g. [30]; the definition is also recalled in (8) below.

Given a subset $E' \subset E$, consider the subset $\mathrm{Conf}(E, E')$ of those configurations whose all particles lie in E'; in Proposition 2.3 below, we shall see that under some additional assumptions the restriction of a determinantal point process onto $\mathrm{Conf}(E, E')$ is again determinantal.

Our main example, the measure $\mathbb{B}^{(s)}$ of (25), is defined on the space of configurations on $(0, +\infty)$. Almost every configuration is infinite and bounded according to $\mathbb{B}^{(s)}$; the particles accumulate at zero. If one takes $R > 0$ and requires all particles to lie in $(0, R)$, then the induced measure of $\mathbb{B}^{(s)}$ on the resulting subset of configurations is finite, and, after normalization, determinantal. As R goes to infinity, the measure of the subset $\mathrm{Conf}((0, +\infty); (0, R))$ grows, and the measure of the space of all configurations is infinite.

Our general construction will similarly exhaust E by subsets E_n in such a way that the weight of $\mathrm{Conf}(E; E_n)$ is positive and finite, and the normalized restriction of our infinite determinantal measure onto the subset $\mathrm{Conf}(E; E_n)$ is determinantal. A simple example is given by "infinite orthogonal polynomial ensembles", see (3) below. The measure $\mathbb{B}^{(s)}$ is a scaling limit of such ensembles. We proceed to precise formulations.

1.2. **Construction of infinite determinantal measures.** Let E be a locally compact complete metric space, and let $\mathrm{Conf}(E)$ be the space of configurations on E endowed with the natural Borel structure (see, e.g., [11], [30]).

Given a Borel subset $E' \subset E$, we let $\mathrm{Conf}(E, E')$ be the subspace of configurations all whose particles lie in E'.

Given a measure $\mathbb{B}$ on a set X and a measurable subset $Y \subset X$ such that $0 < \mathbb{B}(Y) < +\infty$, we let $\mathbb{B}|_Y$ stand for the restriction of the measure $\mathbb{B}$ onto the subset Y.

An *infinite determinantal measure* is a σ-finite Borel measure $\mathbb{B}$ on $\mathrm{Conf}(E)$ admitting a filtration of the space E by Borel subsets E_n, $n \in \mathbb{N}$:

$$E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots \,, \quad \bigcup_{n=1}^{\infty} E_n = E$$

such that for any $n \in \mathbb{N}$ we have

(1) $0 < \mathbb{B}\left(\mathrm{Conf}(E, E_n)\right) < +\infty$;
(2) the normalized restriction

$$\frac{\mathbb{B}\big|_{\mathrm{Conf}(E,E_n)}}{\mathbb{B}\left(\mathrm{Conf}(E, E_n)\right)}$$

is a determinantal measure;
(3) $\mathbb{B}\left(\mathrm{Conf}(E) \setminus \bigcup\limits_{n=1}^{\infty} (\mathrm{Conf}(E, E_n)\right) = 0\,.$

Let μ be a σ-finite Borel measure on E. By the Macchì-Soshnikov Theorem, under some additional assumptions, a determinantal measure can be assigned to an operator of orthogonal projection, or, in other words, to a closed subspace of $L_2(E, \mu)$. In a similar way, an infinite determinantal measure will be assigned to a subspace H of *locally* square-integrable functions. For example, for infinite analogues of orthogonal polynomial ensembles, H is the subspace of weighted polynomials, see Subsection 1.3 below.

Let $L_{2,\mathrm{loc}}(E,\mu)$ be the space of measurable functions on E, locally square integrable with respect to μ, let $\mathscr{I}_1(E,\mu)$ be the space of trace-class operators in $L_2(E,\mu)$ and let $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be the space of operators on $L_2(E,\mu)$ that are locally of trace class (precise definitions are recalled in Section 2).

Let $H \subset L_{2,\mathrm{loc}}(E,\mu)$ be a linear subspace. If $E' \subset E$ is a Borel subset such that $\chi_{E'}H$ is a closed subspace of $L_2(E,\mu)$, then we denote by $\Pi^{E'}$ the operator of orthogonal projection onto the subspace $\chi_{E'}H \subset L_2(E,\mu)$. We now fix a Borel subset $E_0 \subset E$; informally, E_0 is the set where the particles accumulate. We impose the following assumption on E_0 and H.

Assumption 1. (1) *For any bounded Borel set $B \subset E$, the space $\chi_{E_0\cup B}H$ is a closed subspace of $L_2(E,\mu)$;*
(2) *For any bounded Borel set $B \subset E \setminus E_0$, we have*

$$\Pi^{E_0\cup B} \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu), \quad \chi_B \Pi^{E_0\cup B}\chi_B \in \mathscr{I}_1(E,\mu); \tag{1}$$

(3) *If $\varphi \in H$ satisfies $\chi_{E_0}\varphi = 0$, then $\varphi = 0$.*

Theorem 1.1. *Let E be a locally compact complete metric space, and let μ be a σ-finite Borel measure on E. If a subspace $H \subset L_{2,\mathrm{loc}}(E,\mu)$ and a Borel subset $E_0 \subset E$ satisfy Assumption 1, then there exists a σ-finite Borel measure $\mathbb{B}$ on $\mathrm{Conf}(E)$ such that*

(1) *$\mathbb{B}$-almost every configuration has at most finitely many particles outside of E_0;*
(2) *for any bounded Borel (possibly empty) subset $B \subset E \setminus E_0$ we have $0 < \mathbb{B}(\mathrm{Conf}(E; E_0\cup B)) < +\infty$ and*

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E;E_0\cup B)}}{\mathbb{B}(\mathrm{Conf}(E;E_0\cup B))} = \mathbb{P}_{\Pi^{E_0\cup B}}.$$

The requirements (1) and (2) determine the measure $\mathbb{B}$ uniquely up to multiplication by a positive constant.

We denote $\mathbf{B}(H,E_0)$ the one-dimensional cone of nonzero infinite determinantal measures induced by H and E_0, and, slightly abusing notation, we write $\mathbb{B} = \mathbb{B}(H,E_0)$ for a representative of the cone.

Remark. If B is a bounded set, then, by definition, we have

$$\mathbf{B}(H,E_0) = \mathbf{B}(H,E_0\cup B).$$

Remark. If $E' \subset E$ is a Borel subset such that $\chi_{E_0\cup E'}$ is a closed subspace in $L_2(E,\mu)$ and the operator $\Pi^{E_0\cup E'}$ of orthogonal projection onto the subspace $\chi_{E_0\cup E'}H$ satisfies

$$\Pi^{E_0\cup E'} \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu), \quad \chi_{E'}\Pi^{E_0\cup E'}\chi_{E'} \in \mathscr{I}_1(E,\mu), \tag{2}$$

then, exhausting E' by bounded sets, from Theorem 1.1 one easily obtains $0 < \mathbb{B}(\mathrm{Conf}(E; E_0 \cup E')) < +\infty$ and

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E;E_0\cup E')}}{\mathbb{B}(\mathrm{Conf}(E; E_0 \cup E'))} = \mathbb{P}_{\Pi^{E_0\cup E'}}.$$

1.3. **Infinite orthogonal polynomial ensembles.** Take an interval $[a, b)$ in $\mathbb{R}$, let $\mathrm{Leb} = dx$ on $[a, b)$ be the Lebesgue measure on $[a, b)$, let ρ be a positive continuous function on $[a, b)$, and assume $\int_a^b \rho(x)dx = +\infty$. Take $N \in \mathbb{N}$ and endow the set $[a, b]^N$ with the measure

$$\prod_{1\leqslant i,j\leqslant N} (x_i - x_j)^2 \prod_{i=1}^{N} \rho(x_i)dx_i, \tag{3}$$

an infinite analogue of an orthogonal polynomial ensemble.

For any $b_1 \in [a, b)$, the induced measure

$$\prod_{1\leqslant i<j\leqslant N} (x_i - x_j)^2 \prod_{i=1}^{N} \rho(x_i)\chi_{[a,b_1]}(x_i)dx_i \tag{4}$$

is finite and, after normalization, can be represented in determinantal form

$$\frac{1}{N!} \det K_N^{\rho,b_1}(x_i, x_j) \prod_{i=1}^{N} \rho(x_i)\chi_{[a,b_1]}(x_i)dx_i,$$

where K_N^{ρ,b_1} is the N-th Christoffel-Darboux kernel formed by orthonormal polynomials corresponding to the "induced" weight $\rho(x)\chi_{[a,b_1]}(x)$.

The infinite measure (3) is thus an infinite determinantal measure corresponding to the subspace $H \subset L_{2,\mathrm{loc}}([a, b), \mathrm{Leb})$ spanned by the functions $x^k\sqrt{\rho(x)}$, $k = 0, \ldots, N-1$, and the subset $E_0 = [a, b_1)$ for an arbitrary $b_1 \in (a, b)$. In the problem of ergodic decomposition of infinite Pickrell measures we shall be especially interested in studying scaling limits of such "infinite orthogonal polynomial ensembles".

1.4. **Organization of the paper.** In the next subsection it is shown that, under certain additional assumptions, an infinite determinantal measure times a multiplicative functional yields after normalization a determinantal point process; for determinantal probability measures this has been established in [8]. We then proceed to our main example of infinite determinantal measures, namely, those obtained as finite-rank perturbations of determinantal point processes. The ergodic decomposition measures of infinite Pickrell measures will be seen to be of this type. In the following subsection it is established that induced processes of an infinite determinantal measure obtained by finite rank perturbation, converge to the unperturbed process.

In Section 2 we recall the definition of determinantal point processes, study the properties of multiplicative functionals of these processes, thus extending the results of [8], and give a sketch of the proof of Theorem 1.1.

In Section 3 we recall the construction, due to Pickrell [21], [22], [23] in the finite case (see also Neretin [16]) and to Borodin and Olshanski [4] in the infinite case, of Pickrell measures on the space of infinite matrices. We then recall the Olshanski-Vershik approach (see [33], [20]) to the Pickrell classification of finite ergodic unitarily-invariant measures on spaces of infinite matrices as well as the result of [7] that implies that the ergodic components of infinite Pickrell measures are almost surely finite; only the decomposing measure is infinite.

In Section 4 we start by considering finite Pickrell measures, for which the ergodic decomposition is given, up to a change of variable, by the Bessel point process of Tracy and Widom [32]. The main result of the paper, Theorem 4.1 , then says that the ergodic decomposition of infinite Pickrell measures is induced by infinite determinantal measures obtained as an explicitly given finite-rank perturbation of the Bessel point processes occurring in the ergodic decomposition of finite Pickrell measures. The scaling limit argument sketched at the end of the section uses precisely the representation, developed in Section 1, of infinite determinantal measures as products of finite determinantal measures and multiplicative functionals.

1.5. **Multiplicative functionals.** Let g be a non-negative measurable function on E, and introduce the *multiplicative functional* $\Psi_g : \mathrm{Conf}(E) \to \mathbb{R}$ by the formula

$$\Psi_g(X) = \prod_{x \in X} g(x). \tag{5}$$

If the infinite product $\prod\limits_{x \in X} g(x)$ absolutely converges to 0 or to ∞, then we set, respectively, $\Psi_g(X) = 0$ or $\Psi_g(X) = \infty$. If the product in the right-hand side fails to converge absolutely, then the multiplicative functional is not defined.

We start with an auxiliary proposition.

Proposition 1.2. *Let a subspace $H \subset L_{2,\mathrm{loc}}(E, \mu)$ and a Borel subset $E_0 \subset E$ satisfy Assumption 1. Let g be a positive bounded measurable function on E such that*

(1) *for any bounded subset $B \subset E$ there exists $\varepsilon_0 = \varepsilon_0(B) > 0$ such that $g(x) > \varepsilon_0$ for all $x \in E_0 \cup B$;*
(2) *we have $\sqrt{g}H \subset L_2(E, \mu)$.*

Then $\sqrt{g}H$ is a closed subspace in $L_2(E, \mu)$.

Under the assumptions of Proposition 1.2, let Π^g be the operator of orthogonal projection onto the closed subspace $\sqrt{g}H$.

Our next aim is to give sufficient conditions for integrability of multiplicative functionals with respect to infinite determinantal measures. We restrict ourselves to the case when the function g only takes values in $(0, 1]$.

Proposition 1.3. *Let a subspace $H \subset L_{2,\mathrm{loc}}(E, \mu)$ and a Borel subset $E_0 \subset E$ satisfy Assumption 1, and let $g\colon E \to (0, 1]$ be a measurable function such that:*

(1) *for any bounded subset $B \subset E$ there exists $\varepsilon_0 = \varepsilon_0(B) > 0$ such that $g(x) > \varepsilon_0$ for all $x \in E_0 \cup B$;*
(2) $\sqrt{g}H \subset L_2(E, \mu)$*;*
(3) $\sqrt{1-g}\chi_{E_0}\Pi^g\chi_{E_0}\sqrt{1-g} \in \mathscr{I}_1(E, \mu)$*;*
(4) $\Pi^g \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$*;*
(5) $\chi_{E\setminus E_0}\Pi^g\chi_{E\setminus E_0} \in \mathscr{I}_1(E, \mu)$.

Then the multiplicative functional Ψ_g is $\mathbb{B}(H, E_0)$-almost surely positive, and we have

(1)
$$\Psi_g \in L_1(\mathrm{Conf}(E), \mathbb{B});$$
(2)
$$\frac{\Psi_g \mathbb{B}}{\int\limits_{\mathrm{Conf}(E)} \Psi_g \, d\mathbb{B}} = \mathbb{P}_{\Pi^g}.$$

Multiplying both sides of the second conclusion of the lemma by $1/\Psi_g = \Psi_{1/g}$, we obtain
$$\mathbb{B} = C \cdot \Psi_{1/g} \cdot \mathbb{P}_{\Pi^g},$$
where C is a positive constant. Our infinite determinantal measure is thus represented as a product of a determinantal probability measure and a convergent non-integrable multiplicative functional.

1.6. **Infinite determinantal measures obtained as finite-rank perturbations of determinantal probability measures.** We now consider infinite determinantal measures induced by subspaces H obtained by adding a finite-dimensional subspace V to a closed subspace $L \subset L_2(E, \mu)$.

Let, therefore, $Q \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$ be the operator of orthogonal projection onto a closed subspace $L \subset L_2(E, \mu)$, let V be a finite-dimensional subspace of $L_{2,\mathrm{loc}}(E, \mu)$, and set $H = L + V$. Let $E_0 \subset E$ be a Borel subset. We shall need the following assumption on L, V and E_0.

Assumption 2. (1) $\chi_{E\setminus E_0}Q\chi_{E\setminus E_0} \in \mathscr{I}_1(E, \mu)$*;*
(2) $\chi_{E_0}V \subset L_2(E, \mu)$*;*

(3) *if* $\varphi \in V$ *satisfies* $\chi_{E_0}\varphi \in \chi_{E_0}L$, *then* $\varphi = 0$;
(4) *if* $\varphi \in L$ *satisfies* $\chi_{E_0}\varphi = 0$, *then* $\varphi = 0$.

Proposition 1.4. *If* L, V *and* E_0 *satisfy Assumption* 2 *then the subspace* $H = L + V$ *and* E_0 *satisfy Assumption* 1.

In particular, for any bounded Borel subset B, the subspace $\chi_{E_0 \cup B}L$ is closed, as one sees by taking $E' = E_0 \cup B$ in the following clear

Proposition 1.5. *Let* $Q \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$ *be the operator of orthogonal projection onto a closed subspace* $L \in L_2(E, \mu)$. *Let* $E' \subset E$ *be a Borel subset such that* $\chi_{E'}Q\chi_{E'} \in \mathscr{I}_1(E, \mu)$ *and that for any function* $\varphi \in L$, *the equality* $\chi_{E'}\varphi = 0$ *implies* $\varphi = 0$. *Then the subspace* $\chi_{E'}L$ *is closed in* $L_2(E, \mu)$.

The subspace H and the Borel subset E_0 therefore define an infinite determinantal measure $\mathbb{B} = \mathbb{B}(H, E_0)$. We now adapt the formulation of Proposition 1.3 to this particular case.

Proposition 1.6. *Let* L, V, *and* E_0 *satisfy Assumption* 2, *let* $\mathbb{B}$ *be the corresponding infinite determinantal measure, and let* $g\colon E \to (0, 1]$ *be a positive measurable function. If* $\sqrt{1-g}Q\sqrt{1-g} \in \mathscr{I}_1(E, \mu)$, *then the multiplicative functional* Ψ_g *is* $\mathbb{B}$*-almost surely well-defined and positive.*

If, additionally, we assume

(1) $\sqrt{g}V \subset L_2(E, \mu)$;
(2) *for any bounded subset* $B \subset E$ *there exists* $\varepsilon_0 = \varepsilon_0(B) > 0$ *such that* $g(x) > \varepsilon_0$ *for all* $x \in E_0 \cup B$,

then we have

(1)
$$\Psi_g \in L_1(\mathrm{Conf}(E), \mathbb{B});$$
(2)
$$\frac{\Psi_g \mathbb{B}}{\displaystyle\int\limits_{\mathrm{Conf}(E)} \Psi_g \, d\mathbb{B}} = \mathbb{P}_{\Pi^g},$$

where, as before, Π^g *is the operator of orthogonal projection onto the closed subspace* $\sqrt{g}H$.

Remark. The subspace $\sqrt{g}H$ is closed by Proposition 1.2.

1.7. **Convergence of approximating kernels.** Our next aim is to show that, under certain additional assumptions, if a sequence g_n of measurable functions converges to 1, then the operators Π^{g_n} considered in Proposition 1.6 converge to Q in $\mathscr{I}_{1,\mathrm{loc}}(E, \mu)$.

Given two closed subspaces H_1, H_2 in $L_2(E,\mu)$, let $\alpha(H_1,H_2)$ be the angle between H_1 and H_2, defined as the infimum of angles between all nonzero vectors in H_1 and H_2; recall that if one of the subspaces has finite dimension, then the infimum is achieved.

Proposition 1.7. *Let L, V, and E_0 satisfy Assumption 2, and assume additionally that we have $V \cap L_2(E,\mu) = 0$. Let $g_n : E \to (0,1]$ be a sequence of positive measurable functions such that*

(1) *for all $n \in \mathbb{N}$ we have $\sqrt{1-g_n} Q \sqrt{1-g_n} \in \mathscr{I}_1(E,\mu)$;*
(2) *for all $n \in \mathbb{N}$ we have $\sqrt{g_n} V \subset L_2(E,\mu)$;*
(3) *there exists $\alpha_0 > 0$ such that for all n we have*

$$\alpha(\sqrt{g_n} H, \sqrt{g_n} V) \geq \alpha_0;$$

(4) *for any bounded $B \subset E$ we have*

$$\inf_{n\in\mathbb{N}, x\in E_0\cup B} g_n(x) > 0;$$

$$\lim_{n\to\infty} \sup_{x\in E_0\cup B} |g_n(x) - 1| = 0.$$

Then, as $n \to \infty$, we have

$$\Pi^{g_n} \to Q \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu).$$

Using the second remark after Theorem 1.1, one can extend Proposition 1.7 also to nonnegative functions that admit zero values. Here we restrict ourselves to characteristic functions of the form $\chi_{E_0\cup B}$ with B bounded, in which case we have the following

Corollary 1.8. *Let B_n be an increasing sequence of bounded Borel sets exhausting $E \setminus E_0$. If there exists $\alpha_0 > 0$ such that for all n we have*

$$\alpha(\chi_{E_0\cup B_n} H, \chi_{E_0\cup B_n} V) \geq \alpha_0,$$

then

$$\Pi^{E_0\cup B_n} \to Q \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu).$$

Informally, Corollary 1.8 means that, as n grows, the induced processes of our determinantal measure on subsets $\mathrm{Conf}(E; E_0 \cup B_n)$ converge to the "unperturbed" determinantal point process $\mathbb{P}_Q$.

2. Multiplicative Functionals of Determinantal Point Processes

2.1. **Locally integrable functions and locally trace class operators.** Recall that $L_{2,\mathrm{loc}}(E,\mu)$ is the space of all measurable functions $f : E \to \mathbb{C}$

such that for any bounded subset $B \subset E$ we have

$$\int\limits_B |f|^2 d\mu < +\infty. \tag{6}$$

Choosing an exhausting family B_n of bounded sets (for instance, balls of radius tending to infinity) and using (6) with $B = B_n$, we endow the space $L_{2,\text{loc}}(E, \mu)$ with a countable family of seminorms which turns it into a complete separable metric space; the topology thus defined does not, of course, depend on the specific choice of the exhausting family.

Let $\mathscr{I}_1(E, \mu)$ be the ideal of trace class operators $\widetilde{K}: L_2(E, \mu) \to L_2(E, \mu)$ (see volume 1 of [26] for the precise definition); the symbol $||\widetilde{K}||_{\mathscr{I}_1}$ will stand for the $\mathscr{I}_1$-norm of the operator $\widetilde{K}$. Let $\mathscr{I}_2(E, \mu)$ be the ideal of Hilbert-Schmidt operators $\widetilde{K}: L_2(E, \mu) \to L_2(E, \mu)$; the symbol $||\widetilde{K}||_{\mathscr{I}_2}$ will stand for the $\mathscr{I}_2$-norm of the operator $\widetilde{K}$.

Let $\mathscr{I}_{1,\text{loc}}(E, \mu)$ be the space of operators $K: L_2(E, \mu) \to L_2(E, \mu)$ such that for any bounded Borel subset $B \subset E$ we have

$$\chi_B K \chi_B \in \mathscr{I}_1(E, \mu).$$

Again, we endow the space $\mathscr{I}_{1,\text{loc}}(E, \mu)$ with a countable family of seminorms

$$||\chi_B K \chi_B||_{\mathscr{I}_1} \tag{7}$$

where, as before, B runs through an exhausting family B_n of bounded sets.

2.2. **Determinantal Point Processes.** A Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ is called *determinantal* if there exists an operator $K \in \mathscr{I}_{1,\text{loc}}(E, \mu)$ such that for any bounded measurable function g, for which $g - 1$ is supported in a bounded set B, we have

$$\mathbb{E}_{\mathbb{P}} \Psi_g = \det\biggl(1 + (g-1)K\chi_B\biggr). \tag{8}$$

The Fredholm determinant in (8) is well-defined since $K \in \mathscr{I}_{1,\text{loc}}(E, \mu)$. The equation (8) determines the measure $\mathbb{P}$ uniquely. If, for a bounded Borel set $B \subset E$, we let $\#_B: \mathrm{Conf}(E) \to \mathbb{N} \cup \{0\}$ be the function that to a configuration assigns the number of its particles belonging to B, then, for any pairwise disjoint bounded Borel sets $B_1, \ldots, B_l \subset E$ and any $z_1, \ldots, z_l \in \mathbb{C}$ from (8) we have $\mathbb{E}_{\mathbb{P}} z_1^{\#_{B_1}} \cdots z_l^{\#_{B_l}} = \det\biggl(1 + \sum\limits_{j=1}^{l} (z_j - 1)\chi_{B_j} K \chi_{\sqcup_i B_i}\biggr)$.

For further results and background on determinantal point processes, see e.g. [2], [9], [12], [13], [14], [27], [28], [29], [30].

In what follows we suppose that K belongs to $\mathscr{I}_{1,\text{loc}}(E, \mu)$, and denote the corresponding determinantal measure by $\mathbb{P}_K$. Note that $\mathbb{P}_K$ is uniquely defined by K, but different operators may yield the same measure. By the Macchì—Soshnikov theorem [15], [30], any Hermitian positive contraction that belongs to the class $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ defines a determinantal point process.

2.3. **Multiplicative functionals.** At the centre of the construction of infinite determinantal measures lies the result of [8] that can informally be summarized as follows: a determinantal measure times a multiplicative functional is again a determinantal measure. In other words, if $\mathbb{P}_K$ is a determinantal measure on $\mathrm{Conf}(E)$ induced by the operator K on $L_2(E,\mu)$, then, under certain additional assumptions, it is shown in [8] that the measure $\Psi_g\mathbb{P}_K$ after normalization yields a determinantal measure.

It is required in [8] that the operator $(g-1)K$ be of trace class; this assumption is too restrictive for our purposes, and in Propositions 2.1 and 2.5 we shall now formulate two more convenient versions of Proposition 1 in [8].

As before, let g be a non-negative measurable function on E. If the operator $1+(g-1)K$ is invertible, then we set

$$\mathfrak{B}(g,K)=gK(1+(g-1)K)^{-1}, \qquad \widetilde{\mathfrak{B}}(g,K)=\sqrt{g}K(1+(g-1)K)^{-1}\sqrt{g}.$$

By definition, $\mathfrak{B}(g,K),\widetilde{\mathfrak{B}}(g,K)\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ since $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$, and, if K is self-adjoint, then so is $\widetilde{\mathfrak{B}}(g,K)$.

In the case when K is self-adjoint, the following proposition generalizes Proposition 1 in [8].

Proposition 2.1. *Let $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be a self-adjoint positive contraction, and let $\mathbb{P}_K$ be the corresponding determinantal measure on $\mathrm{Conf}(E)$. Let g be a nonnegative bounded measurable function on E such that*

$$\sqrt{g-1}K\sqrt{g-1}\in\mathscr{I}_1(E,\mu) \tag{9}$$

and that the operator $1+(g-1)K$ is invertible. Then

(1) *we have $\Psi_g\in L_1(\mathrm{Conf}(E),\mathbb{P}_K)$ and*

$$\int\Psi_g\,d\mathbb{P}_K=\det\Big(1+\sqrt{g-1}K\sqrt{g-1}\Big)>0;$$

(2) *the operators $\mathfrak{B}(g,K),\widetilde{\mathfrak{B}}(g,K)$ induce on $\mathrm{Conf}(E)$ a determinantal measure $\mathbb{P}_{\mathfrak{B}(g,K)}=\mathbb{P}_{\widetilde{\mathfrak{B}}(g,K)}$ satisfying*

$$\mathbb{P}_{\mathfrak{B}(g,K)}=\frac{\Psi_g\mathbb{P}_K}{\int\limits_{\mathrm{Conf}(E)}\Psi_g\,d\mathbb{P}_K}. \tag{10}$$

Remark. Since (9) holds and K is self-adjoint, the operator $1+(g-1)K$ is invertible if and only if the operator $1+\sqrt{g-1}K\sqrt{g-1}$ is invertible.

If Q is a projection operator, then the operator $\tilde{\mathfrak{B}}(g, Q)$ admits the following description.

Proposition 2.2. *Let $L \subset L_2(E, \mu)$ be a closed subspace, and let Q be the operator of orthogonal projection onto L. Let g be a bounded measurable function such that the operator $1+(g-1)Q$ is invertible. Then the operator $\tilde{\mathfrak{B}}(g, Q)$ is the operator of orthogonal projection onto the closure of the subspace $\sqrt{g}L$.*

We now consider the particular case when g is a characteristic function of a Borel subset. In much the same way as before, if $E' \subset E$ is a Borel subset such that the subspace $\chi_{E'}L$ is closed (recall that a sufficient condition for that is provided in Proposition 1.5), then we set $Q^{E'}$ to be the operator of orthogonal projection onto the closed subspace $\chi_{E'}L$.

Propositions 2.5, 2.1 now yield the following

Corollary 2.3. *Let $Q \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$ be the operator of orthogonal projection onto a closed subspace $L \in L_2(E, \mu)$. Let $E' \subset E$ be a Borel subset such that $\chi_{E'}Q\chi_{E'} \in \mathscr{I}_1(E, \mu)$. Then*

$$\mathbb{P}_Q(\mathrm{Conf}(E, E')) = \det(1 - \chi_{E \setminus E'}Q\chi_{E \setminus E'}).$$

Assume, additionally, that for any function $\varphi \in L$, the equality $\chi_{E'}\varphi = 0$ implies $\varphi = 0$. Then the subspace $\chi_{E'}L$ is closed, and we have

$$\mathbb{P}_Q(\mathrm{Conf}(E, E')) > 0, \ Q^{E'} \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu),$$

and

$$\frac{\mathbb{P}_Q|_{\mathrm{Conf}(E,E')}}{\mathbb{P}_Q(\mathrm{Conf}(E, E'))} = \mathbb{P}_{Q^{E'}}.$$

The induced measure of a determinantal measure onto the subset of configurations all whose particles lie in E' is thus again a determinantal measure. In the case of a discrete phase space, related induced processes were considered by Lyons [12] and by Borodin and Rains [5].

Corollary 2.3 implies Theorem 1.1.

2.4. **The algebra $\mathscr{I}_\xi$.** To prove Proposition 2.1, we use a variant of the Hilbert-Carleman regularization of Fredholm determinants, namely, we consider a slightly more general algebra of operators K for which the trace $\mathrm{tr}\, K$ and the Fredholm determinant $\det(1+K)$ can be defined and shown to have the usual properties. The algebra $\mathscr{I}_\xi(E, \mu)$ is a modification of the algebra $\mathcal{L}_{1|2}(H)$ introduced by Borodin, Okounkov and Olshanski [3] and used also by Olshanski in [17]. We proceed to precise formulations.

Take a countable partition ξ of our space E into disjoint bounded measurable sets E_n, $n \in \mathbb{N}$. Introduce the sets

$$\{\xi > n\} = \bigcup_{k=n+1}^{\infty} E_k;\ \{\xi < n\} = \bigcup_{k=1}^{n-1} E_k. \tag{11}$$

Informally, ξ is considered as a random variable taking integer values.

The subspace

$$\mathscr{I}_\xi(E,\mu) \subset \mathscr{I}_{1,loc}(E,\mu)$$

is now defined as follows: an operator $K \in \mathscr{I}_{1,loc}(E,\mu)$ belongs to $\mathscr{I}_\xi(E,\mu)$ if

(1) $K \in \mathscr{I}_2(E,\mu)$;
(2) $\sum\limits_{n=1}^{\infty} ||\chi_{E_n} K \chi_{E_n}||_{\mathscr{I}_1} < +\infty$.

The space $\mathscr{I}_\xi(E,\mu)$ is normed by the formula

$$||K||_{\mathscr{I}_\xi} = ||K||_{\mathscr{I}_2} + \sum_{n=1}^{\infty} ||\chi_{E_n} K \chi_{E_n}||_{\mathscr{I}_1}.$$

By definition, the space $\mathscr{I}_\xi(E,\mu)$ is an algebra. For $K \in \mathscr{I}_\xi(E,\mu)$, the Fredholm determinant $\det(1+K)$ is defined by the formula

$$\det(1+K) = \det\left((1+K)\exp(-K)\right)\exp\left(\sum_{n=1}^{\infty} \operatorname{tr}(\chi_{E_n} K \chi_{E_n})\right). \tag{12}$$

The right-hand side of (12) is well-defined since $(1+K)\exp(-K) \in \mathscr{I}_1$ for any $K \in \mathscr{I}_2$.

For $K_1, K_2 \in \mathscr{I}_\xi$, we clearly have

$$\det((1+K_1)(1+K_2)) = \det(1+K_1)\det(1+K_2).$$

From the definitions we now immediately obtain

Proposition 2.4. *If $(g-1)K \in \mathscr{I}_\xi(E,\mu)$, then $\Psi_g \in L_1(\mathrm{Conf}(E), \mathbb{P}_K)$ and*

$$\mathbb{E}_{\mathbb{P}_K}\Psi_g = \det(1+(g-1)K).$$

The following Proposition is a generalization of Proposition 1 in [8].

Proposition 2.5. *Assume that an operator $K \in \mathscr{I}_{1,loc}(E,\mu)$ induces a determinantal measure $\mathbb{P}_K$ on $\mathrm{Conf}(E)$. Let ξ be a countable measurable partition of E and let g be a nonnegative bounded measurable function on E such that $(g-1)K \in \mathscr{I}_\xi(E,\mu)$ and that the operator $1+(g-1)K$*

is invertible. Then the operators $\mathfrak{B}(g,K), \widetilde{\mathfrak{B}}(g,K)$ *induce on* $\mathrm{Conf}(E)$ *a determinantal measure* $\mathbb{P}_{\mathfrak{B}(g,K)} = \mathbb{P}_{\widetilde{\mathfrak{B}}(g,K)}$ *satisfying*

$$\mathbb{P}_{\mathfrak{B}(g,K)} = \frac{\Psi_g \mathbb{P}_K}{\int\limits_{\mathrm{Conf}(E)} \Psi_g d\mathbb{P}_K}. \tag{13}$$

Indeed, take a bounded measurable function f on E such that

$$(f-1)K \in \mathscr{I}_\xi(E,\mu).$$

We then immediately have

$$\frac{\mathbb{E}_{\mathbb{P}_K}\Psi_f\Psi_g}{\mathbb{E}_{\mathbb{P}_K}\Psi_g} = \det(1+(f-1)\mathfrak{B}(g,K)) = \det(1+(f-1)\tilde{\mathfrak{B}}(g,K)),$$

and the proposition follows.

Observe now that to a nonnegative function g such that (9) holds, one can easily assign a countable partition ξ such that $(g-1)K \in \mathscr{I}_\xi(E,\mu)$. Proposition 2.1 is therefore clear from Proposition 2.5.

3. Unitarily-Invariant Measures on Spaces of Infinite Matrices

3.1. **Pickrell Measures.** Let $\mathrm{Mat}(n,\mathbb{C})$ be the space of $n\times n$ matrices with complex entries:

$$\mathrm{Mat}(n,\mathbb{C}) = \{z=(z_{ij}),\ i=1,\dots,n; j=1,\dots,n\}$$

Let $\mathrm{Leb} = dz$ be the Lebesgue measure on $\mathrm{Mat}(n,\mathbb{C})$.

Following Pickrell [21], take $s \in \mathbb{R}$ and introduce a measure $\widetilde{\mu}_n^{(s)}$ on $\mathrm{Mat}(n,\mathbb{C})$ by the formula

$$\widetilde{\mu}_n^{(s)} = \det(1+z^*z)^{-2n-s}dz.$$

The measure $\widetilde{\mu}_n^{(s)}$ is finite if and only if $s > -1$.

For $n_1 < n$, let

$$\pi^n_{n_1} : \mathrm{Mat}(n,\mathbb{C}) \to \mathrm{Mat}(n_1,\mathbb{C})$$

be the natural projection map that to a matrix $z = (z_{ij}), i,j = 1,\dots,n$, assigns its upper left corner, the matrix $\pi^n_{n_1}(z) = (z_{ij}), i,j = 1,\dots,n_1$.

The measures $\widetilde{\mu}_n^{(s)}$ have the property of consistency with respect to the projections $\pi^n_{n_1}$. More precisely, following Borodin and Olshanski [4], p.116, observe that even if the measure $\widetilde{\mu}_n^{(s)}$ is infinite, the fibres of the projection π^n_{n-1} have finite conditional measure as long as $n+s > 0$. The push-forward $(\pi^n_{n-1})_*\widetilde{\mu}^{(s)}_n$ is consequently well-defined, and for any $s \in \mathbb{R}$ and $n > -s$ we have

$$(\pi^n_{n-1})_*\widetilde{\mu}^{(s)}_n = \frac{\pi^{2n-1}(\Gamma(n+s))^2}{\Gamma(2n+s)\cdot\Gamma(2n-1+s)}\widetilde{\mu}^{(s)}_{n-1}. \tag{14}$$

Now let $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ be the space of infinite matrices whose rows and columns are indexed by natural numbers and whose entries are complex:

$$\mathrm{Mat}(\mathbb{N},\mathbb{C}) = \{z = (z_{ij}), i, j \in \mathbb{N}, z_{ij} \in \mathbb{C}\}.$$

Let $\pi^\infty_n : \mathrm{Mat}(\mathbb{N},\mathbb{C}) \to \mathrm{Mat}(n,\mathbb{C})$ be the natural projection map that to an infinite matrix $z \in \mathrm{Mat}(\mathbb{N},\mathbb{C})$ assigns its upper left $n \times n$-“corner”, the matrix $(z_{ij}), i, j = 1, \dots, n$.

Take $s \in \mathbb{R}$ and $n_0 \in \mathbb{N}, n_0 > -s$. The relation (14) and the Kolmogorov Existence Theorem [10] imply (for a detailed presentation, see p. 116 in Borodin and Olshanski [4]) that for any $\lambda > 0$ there exists a unique measure $\mu^{(s,\lambda)}$ on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ such that for any $n > n_0$ we have

$$(\pi^\infty_n)_*\mu^{(s,\lambda)} = \lambda\left(\prod_{l=n_0}^{n}\pi^{-2n}\frac{\Gamma(2l+s)\Gamma(2l-1+s)}{(\Gamma(l+s))^2}\right)\widetilde{\mu}^{(s)}. \tag{15}$$

If $s > -1$, the measures $\mu^{(s,\lambda)}$ are finite, and we let $\mu^{(s)}$ be the probability measure in the family $\mu^{(s,\lambda)}$.

In this case, (14) implies the relation

$$(\pi^\infty_n)_*\mu^{(s)} = \pi^{-n^2}\prod_{l=1}^{n}\frac{\Gamma(2l+s)\Gamma(2l-1+s)}{(\Gamma(l+s))^2}\widetilde{\mu}^{(s)}_n.$$

If $s \leqslant -1$, the measures $\mu^{(s,\lambda)}$ are all infinite. In this case, slightly abusing notation, we shall omit the super-script λ and write $\mu^{(s)}$ for a measure defined up to a multiplicative constant.

Proposition 3.1. *For any $s_1, s_2 \in \mathbb{R}$, $s_1 \neq s_2$, the Pickrell measures $\mu^{(s_1)}$ and $\mu^{(s_2)}$ are mutually singular.*

Proposition 3.1 is obtained from Kakutani's Theorem in the spirit of [4], see also [16].

Let $U(\infty)$ be the infinite unitary group: an infinite matrix $u = (u_{ij})_{i,j\in\mathbb{N}}$ belongs to $U(\infty)$ if there exists a natural number n_0 such that the matrix

$$(u_{ij}), i, j \in [1, n_0]$$

is unitary, while $u_{ii} = 1$ if $i > n_0$ and $u_{ij} = 0$ if $i \neq j$, $\max(i,j) > n_0$.

The group $U(\infty) \times U(\infty)$ acts on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ by multiplication on both sides:

$$T_{(u_1,u_2)}z = u_1 z u_2^{-1}.$$

The Pickrell measures $\mu^{(s)}$ are by definition $U(\infty) \times U(\infty)$-invariant. For the rôle of Pickrell and related mesures in the representation theory of $U(\infty)$, see [18], [19], [20].

Theorem 1 and Corollary 1 in [6] imply that the measures $\mu^{(s)}$ admit an ergodic decomposition. Furthermore, Theorem 1 in [7] implies that for any $s \in \mathbb{R}$ the ergodic components of the measure $\mu^{(s)}$ are almost surely finite. The main result of this note is an explicit description of the ergodic decomposition of the measures $\mu^{(s)}$ for $s \neq -1-2k, k \in \mathbb{N}$; in particular, for $s < -1$ we shall see that the ergodic decomposition is given by an explicitly computed infinite determinantal measure.

3.2. **Classification of ergodic measures.** First, we recall the classification of ergodic probability $U(\infty) \times U(\infty)$-invariant measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$. This classification has been obtained by Pickrell [21], [22]; Vershik [33] and Olshanski and Vershik [20] proposed a different approach to this classification in the case of unitarily-invariant measures on the space of infinite Hermitian matrices, and Rabaoui [24], [25] adapted the Olshanski-Vershik approach to the initial problem of Pickrell. In this note, the Olshanski-Vershik approach is followed as well.

Take $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$, denote $z^{(n)} = \pi_n^{\infty} z$, and let

$$\lambda_1^{(n)} \geqslant \ldots \geqslant \lambda_n^{(n)} \geqslant 0 \tag{16}$$

be the eigenvalues of the matrix

$$\left(z^{(n)}\right)^* z^{(n)},$$

counted with multiplicities, arranged in non-increasing order. To stress dependence on z, we write $\lambda_i^{(n)} = \lambda_i^{(n)}(z)$.

Theorem. (1) *Let η be an ergodic Borel $U(\infty) \times U(\infty)$-invariant probability measure on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$. Then there exist non-negative real numbers*

$$\gamma \geqslant 0, \;\; x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant \ldots \geqslant 0\,,$$

satisfying $\gamma \geqslant \sum_{i=1}^{\infty} x_i$, such that for η-almost every $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ and any $i \in \mathbb{N}$ we have:

$$x_i = \lim_{n\to\infty} \frac{\lambda_i^{(n)}(z)}{n^2}, \;\; \gamma = \lim_{n\to\infty} \frac{\mathrm{tr}\left(z^{(n)}\right)^* z^{(n)}}{n^2}. \tag{17}$$

(2) *Conversely, given non-negative real numbers* $\gamma \geqslant 0,\ \ x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant \ldots \geqslant 0$ *such that*

$$\gamma \geqslant \sum_{i=1}^{\infty} x_i\,,$$

there exists a unique $U(\infty)\times U(\infty)$*-invariant ergodic Borel probability measure* η *on* $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ *such that the relations* (17) *hold for* η*-almost all* $z \in \mathrm{Mat}(\mathbb{N},\mathbb{C})$.

Introduce the *Pickrell set* $\Omega_P \subset \mathbb{R}_+ \times \mathbb{R}_+^{\mathbb{N}}$ by the formula

$$\Omega_P = \left\{\omega = (\gamma, x)\colon x = (x_n),\ n \in \mathbb{N},\ x_n \geqslant x_{n+1} \geqslant 0,\ \gamma \geqslant \sum_{i=1}^{\infty} x_i \right\}.$$

The set Ω_P is, by definition, a closed subset of $\mathbb{R}_+ \times \mathbb{R}_+^{\mathbb{N}}$ endowed with the Tychonoff topology.

By Proposition 3 in [6], the subset of ergodic $U(\infty)\times U(\infty)$-invariant measures is a Borel subset of the space of all Borel probability measures on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ endowed with the natural Borel structure (see, e.g., [1]). Furthermore, if one denotes η_ω the Borel ergodic probability measure corresponding to a point $\omega \in \Omega_P$, $\omega = (\gamma, x)$, then the correspondence

$$\omega \longrightarrow \eta_\omega$$

is a Borel isomorphism of the Pickrell set Ω_P and the set of $U(\infty)\times U(\infty)$-invariant ergodic probability measures on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$.

The Ergodic Decomposition Theorem (Theorem 1 and Corollary 1 of [6]) implies that each Pickrell measure $\mu^{(s)}$, $s \in \mathbb{R}$, induces a unique decomposing measure $\overline{\mu}^{(s)}$ on Ω_P such that we have

$$\mu^{(s)} = \int\limits_{\Omega_P} \eta_\omega \, d\overline{\mu}^{(s)}(\omega)\ . \tag{18}$$

The integral is understood in the usual weak sense, see [6].

For $s > -1$, the measure $\overline{\mu}^{(s)}$ is a probability measure on Ω_P, while for $s \leqslant -1$ the measure $\overline{\mu}^{(s)}$ is infinite.

Set

$$\Omega_P^0 = \{(\gamma, \{x_n\}) \in \Omega_P : x_n > 0 \quad \text{for all } n,\ \gamma = \sum_{n=1}^{\infty} x_n\}.$$

The subset Ω_P^0 is of course not closed in Ω_P.

Introduce a map

$$\mathrm{conf}\colon \Omega_P \to \mathrm{Conf}((0,+\infty))$$

that to a point $\omega \in \Omega_P, \omega = (\gamma, \{x_n\})$ assigns the configuration

$$\mathrm{conf}(\omega) = (x_1, \ldots, x_n, \ldots) \in \mathrm{Conf}((0, +\infty)).$$

The map $\omega \to \mathrm{conf}(\omega)$ is bijective in restriction to the subset Ω_P^0.

Remark. In the definition of the map conf, the "asymptotic eigenvalues" x_n are counted with multiplicities, while, if $x_{n_0} = 0$ for some n_0, then x_{n_0} and all subsequent terms are discarded, and the resulting configuration is finite. We shall see, however, that the complement $\Omega_P \backslash \Omega_P^0$ is $\overline{\mu}^{(s)}$-negligible for all $s \neq -1 - 2k$, $k \in \mathbb{N}$, and, consequently, that, $\overline{\mu}^{(s)}$-almost surely, all configurations are infinite. It will also develop that, $\overline{\mu}^{(s)}$-almost surely, all multiplicities are equal to one.

3.3. **The Bessel point process and finite Pickrell measures.** Consider the half-line $(0, +\infty)$ endowed with the standard Lebesgue measure Leb. Take $s > -1$ and consider the standard Bessel kernel

$$J_s(x, y) = \frac{\sqrt{x} J_{s+1}(\sqrt{x}) J_s(\sqrt{y}) - \sqrt{y} J_{s+1}(\sqrt{y}) J_s(\sqrt{x})}{2(x - y)} \tag{19}$$

(see, e.g., page 295 in Tracy and Widom [32]). The kernel J_s induces on $L_2((0, +\infty), \mathrm{Leb})$ the operator of orthogonal projection onto the subspace of functions whose Hankel transform is supported in $[0, 1]$ (see [32]). Setting $x_1 = 4/x,\ x_2 = 4/y$ yields a kernel $K^{(s)}$ given by the formula

(20)

$$K^{(s)}(x_1, x_2) = \frac{J_s\left(\frac{2}{\sqrt{x_1}}\right) \frac{1}{\sqrt{x_2}} J_{s+1}\left(\frac{2}{\sqrt{x_2}}\right) - J_s\left(\frac{2}{\sqrt{x_2}}\right) \frac{1}{\sqrt{x_1}} J_{s+1}\left(\frac{2}{\sqrt{x_1}}\right)}{x_1 - x_2},$$
$$x_1 > 0,\ x_2 > 0\,.$$

(recall here that a change of variables $u_1 = \rho(v_1)$, $u_2 = \rho(v_2)$ transforms a kernel $K(u_1, u_2)$ to a kernel of the form $K(\rho(v_1), \rho(v_2))(\sqrt{\rho'(v_1)\rho'(v_2)})$).

The kernel $K^{(s)}$ induces on the space $L_2((0, +\infty), \mathrm{Leb})$ a locally trace class operator of orthogonal projection, for which, slightly abusing notation, we keep the symbol $K^{(s)}$; by the Macchì-Soshnikov Theorem, the operator $K^{(s)}$ induces a determinantal measure $\mathbb{P}_{K^{(s)}}$ on $\mathrm{Conf}((0, +\infty))$. The determinantal measure $\mathbb{P}_{K^{(s)}}$ is precisely the decomposing measure for the Pickrell measure $\mu^{(s)}$, as is shown by the following

Proposition 3.2. *Let $s > -1$. Then $\overline{\mu}^{(s)}(\Omega_P^0) = 1$ and the $\overline{\mu}^{(s)}$-almost sure bijection $\omega \to \mathrm{conf}(\omega)$ identifies the measure $\overline{\mu}^{(s)}$ with the determinantal measure $\mathbb{P}_{K^{(s)}}$.*

Sketch of proof of Proposition 3.2. Take $s > -1$. Let $P_n^{(s)}(u)$ be the standard Jacobi orthogonal polynomials on the interval $[-1, 1]$ corresponding to the weight $(1-u)^s$ (recall here that Jacobi polynomials $P_n^{(s_1,s_2)}$ are usually defined as polynomials on the interval $[-1,1]$ orthogonal with the weight $(1-u)^{s_1}(1+u)^{s_2}$ and satisfying $P_n^{(s_1,s_2)} = \Gamma(n+s_1+1)/\Gamma(n+1)\Gamma(s_1+1)$, see, e.g. (4.1.1) in Szegö [31]; it would thus have been more precise to write $P_n^{(s,0)}(u)$, but, since we will never need the second parameter s_2, for brevity we omit the second superscript) .

Following Pickrell, to a matrix $z \in \mathrm{Mat}(n,\mathbb{C})$ assign the collection $(\lambda_1(z),\ldots,\lambda_n(z))$ of the eigenvalues of the matrix z^*z arranged in non-increasing order (cf. (16)). The *radial part* $\mathfrak{r}^{(n,s)}$ of the Pickrell measure $\mu_n^{(s)}$ is now defined as the push-forward of the measure $\mu_n^{(s)}$ under the map

$$z \to (\lambda_1(z),\ldots,\lambda_n(z))\,.$$

The radial part of the Pickrell measure has determinantal form:

$$d\mathfrak{r}^{(n,s)}(\lambda) = \frac{1}{n!}\det K_n^{(s)}\left(\lambda_i,\lambda_j\right)\prod_{i=1}^{n} d\lambda_i, \quad \lambda_i > 0\,.$$

where

$$K_n^{(s)}\left(\lambda_1,\lambda_2\right) = \frac{n(n+s)}{(2n+s)(1+\lambda_1)^{s/2}(1+\lambda_2)^{s/2}}\times \tag{21}$$
$$\times\frac{P_n^{(s)}\left(\frac{\lambda_1-1}{\lambda_1+1}\right)P_{n-1}^{(s)}\left(\frac{\lambda_2-1}{\lambda_2+1}\right) - P_n^{(s)}\left(\frac{\lambda_2-1}{\lambda_2+1}\right)P_{n-1}^{(s)}\left(\frac{\lambda_1-1}{\lambda_1+1}\right)}{\lambda_1-\lambda_2}.$$

The change of variables $u_i = \dfrac{\lambda_i-1}{\lambda_i+1}$, $i=1,\ldots,n$, reduces $K_n^{(s)}$ to the Christoffel-Darboux kernel for the Jacobi orthogonal ensemble with weight $(1-u)^s$.

Introducing the scaling $\lambda_i = n^2 x_i$, taking $n\to\infty$ and using the classical Heine-Mehler asymptotics for Jacobi orthogonal polynomials (see, e.g., Szegö [31]), one finds

$$\lim_{n\to\infty} n^2 K_n^{(s)}\left(n^2x_1,n^2x_2\right) = K^{(s)}(x_1,x_2)\,,$$

convergence being uniform on compact subsets of $(0,+\infty)$. To prove that $\overline{\mu}^{(s)}(\Omega_P^0) = 1$, the method of Section 7 in Borodin and Olshanski [4] is adapted to our situation.

3.4. **A recurrence relation for Bessel point proceses.** The following observation motivates the construction of the next section. Given a finite family of functions $f_1,\ldots,f_N$ on the real line, let $\mathrm{span}(f_1,\ldots,f_N)$ stand for the vector space these functions span. For an arbitrary function ρ of the real variable u and any $N \in \mathbb{N}$ we clearly have

$$\text{span}\left(\rho, u\rho, \ldots, u^N\rho\right) = \mathbb{R}\rho \oplus \oplus \text{span}\left((1-u)\rho, u(1-u)\rho, , \ldots, u^{N-1}(1-u)\rho\right). \tag{22}$$

Assume additionally that the N-th orthogonal polynomial ensemble with weight ρ^2 is well-defined. In this case the $N-1$-th orthogonal polynomial ensemble with weight $\rho^2(1-u)^2$ is also well-defined, and the equality (22) states that the space of the first $N+1$ normalized orthogonal polynomials with weight ρ^2 is a rank one perturbation of the space of the first N normalized orthogonal polynomials with weight $(1-u)^2\rho^2$.

Take $s \in \mathbb{R}$ and set $\rho(u) = (1-u)^{s/2}$. Take $N \in \mathbb{N}$ and rewrite (23) in this particular case:

$$\text{span}\left((1-u)^{s/2}, \ldots, (1-u)^{s/2}u^N\right) = \mathbb{R}(1-u)^{s/2} \oplus \oplus \text{span}\left((1-u)^{(s+2)/2}, \ldots, (1-u)^{(s+2)/2}u^{N-1}\right). \tag{23}$$

If $s > -1$, then (23) states that the space of the first $N+1$ normalized Jacobi polynomials with weight $(1-u)^s$ is a rank one perturbation of the space of the first N normalized Jacobi polynomials with weight $(1-u)^{s+2}$.

A similar statement holds true for the Bessel kernel: using the recurrence relation $J_{s+1}(x) = \frac{2s}{x}J_s(x) - J_{s-1}(x)$ for Bessel functions, one easily obtains the recurrence relation

$$J_s(x,y) = J_{s+2}(x,y) + \frac{s+1}{\sqrt{xy}} J_{s+1}(\sqrt{x})J_{s+1}(\sqrt{y}) \tag{24}$$

for the Bessel kernels: the Bessel kernel with parameter s is thus a rank one perturbation of the Bessel kernel with parameter $s+2$.

For ergodic decomposition measures of infinite Pickrell measures we shall now give a similar description in terms of infinite determinantal measures obtained as finite-rank perturbations of Bessel point processes.

4. Ergodic decomposition of infinite Pickrell measures

Now take $s < -1$, $s \neq -1-2k$, $k \in \mathbb{N}$. Let n_s be such that

$$\frac{s}{2} + n_s \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Introduce a finite-dimensional subspace $V^{(s)} \subset L_{2,loc}((0,+\infty), \text{Leb})$ by the formula

$$V^{(s)} = \text{span}\left(x^{-s/2-1}, \ldots, x^{-s/2-n_s}\right).$$

For $s' > -1$, let $L^{(s')} \subset L_2((0,+\infty), \text{Leb})$ be the range of the operator $K^{s'}$, and for $s < -1$, $s \neq -1-2k$, $k \in \mathbb{N}$, introduce a subspace $H^{(s)}$ of

$L_{2,\mathrm{loc}}((0,+\infty),\mathrm{Leb})$ by the formula

$$H^{(s)} = L^{(s+2n_s)} + V^{(s)}.$$

Using Proposition 1.4, one easily checks that if $R > 0$ is big enough, then the subspace $H^{(s)} \subset L_2((0,+\infty),\mathrm{Leb})$ and the subset $E_0 = (0,R)$ satisfy Assumption 1. Let

$$\mathbb{B}^{(s)} = \mathbb{B}(H^{(s)}, (0,R)) \tag{25}$$

be the corresponding infinite determinantal measure (which, by definition, does not depend on the specific choice of a big enough R).

The ergodic decomposition of infinite Pickrell measures is now given by the following

Theorem 4.1. *Let $s < -1$, $s \neq -1-2k$, $k \in \mathbb{N}$, and let $\overline{\mu}^{(s)}$ be the decomposing measure, defined by (18), of the Pickrell measure $\mu^{(s)}$. Then*

(1) $\overline{\mu}^{(s)}(\Omega_P \setminus \Omega_P^0) = 0$;
(2) *the $\overline{\mu}^{(s)}$-almost sure bijection $\omega \to \mathrm{conf}(\omega)$ identifies $\overline{\mu}^{(s)}$ with the infinite determinantal measure $\mathbb{B}^{(s)}$.*

Take $R > 0$ and set

$$\Omega_P^0(R) = \{\omega \in \Omega_P^0 : x_1 \leq R\}.$$

Set $L_R^{(s)} = \chi_{(0,R)} H^{(s)}$; the subspace $L_R^{(s)}$ is closed if $R > 0$ is big enough, and we let $Q_R^{(s)}$ be the corresponding operator of orthogonal projection. By Proposition 1.7, we have $Q_R^{(s)} \to K^{(s+2n_s)}$ in $\mathscr{I}_{1,\mathrm{loc}}((0,+\infty),\mathrm{Leb})$ as $R \to \infty$. Theorem 4.1 together with Theorem 1.1 implies

Corollary 4.2. *If R is big enough, then $0 < \overline{\mu}^{(s)}(\Omega_P^0(R)) < +\infty$ and the $\overline{\mu}^{(s)}$-almost sure bijection $\omega \to \mathrm{conf}(\omega)$ identifies the normalized restriction of the measure $\overline{\mu}^{(s)}$ to the subset $\Omega_P^0(R)$ with the determinantal measure $\mathbb{P}_{Q_R^{(s)}}$.*

We now give an explicit representation of the measure $\overline{\mu}^{(s)}$ as a product of a determinantal measure and a multiplicative functional. Take $\alpha > 0$ and let $L^{(s,\alpha)} = \exp^{-\alpha x} H^{(s)}$. By definition, $L^{(s,\alpha)} \subset L_2\left((0,+\infty),\mathrm{Leb}\right)$. Let $Q^{(s,\alpha)}$ be the operator of orthogonal projection onto the subspace $L^{(s,\alpha)}$. By definition, $Q^{(s,\alpha)} \in \mathscr{I}_{1,\mathrm{loc}}\left((0,+\infty),\mathrm{Leb}\right)$, and, by the Macchì-Soshnikov Theorem, the operator $Q^{(s,\alpha)}$ induces a determinantal measure $\mathbb{P}_{Q^{(s,\alpha)}}$ on $\mathrm{Conf}\big((0,+\infty)\big)$.

Given a configuration $X \in \mathrm{Conf}\big((0,+\infty)\big)$, let $S(X) = \sum\limits_{x\in X} x$ be the function that to a configuration assigns the (possibly infinite) sum of all its particles. By construction, the measure $\mathbb{P}_{Q^{(s,\alpha)}}$ is supported on the set $\{X\colon S(X) < +\infty\}$. Theorem 4.1 and Propositions 1.3, 1.6 now imply

Corollary 4.3. *For any $\alpha > 0$, the $\overline{\mu}^{(s)}$-almost sure bijection $\omega \to \mathrm{conf}(\omega)$ identifies the measure $\overline{\mu}^{(s)}$ wtih a measure of the form $Ce^{\alpha\, S(X)}\mathbb{P}_{Q(s,\alpha)}$, where C is a positive constant.*

The proof of Theorem 4.1 starts, again, with the computation of the radial part of the infinite Pickrell measure; changing variables by the formula

$$u_i = \frac{\lambda_i - 1}{\lambda_i + 1},\ i = 1, \ldots, n,$$

one arrives at an "infinite orthogonal polynomial ensemble"(cf. (3)) of the form

$$\prod_{i<j}(u_i - u_j)^2 \prod_i (1 - u_i)^s. \tag{26}$$

By definition, the measure (26) is an infinite determinantal measure obtained by perturbing the closed subspace

$$\mathrm{span}\left((1-u)^{(s+2n_s)/2}, \ldots, (1-u)^{(s+2n_s)/2}u^{N-n_s-1}\right) \subset L_2([-1,1], \mathrm{Leb})$$

by the finite-dimensional subspace

$$\mathrm{span}\left((1-u)^{s/2}, \ldots, (1-u)^{(s+2n_s-2)/2}\right) \subset L_{2,\mathrm{loc}}([-1,1], \mathrm{Leb}).$$

The next step is to take the scaling limit of these infinite determinantal measures. This is achieved, with the use of Propositions 1.3 and 1.6, by taking the product with a suitably chosen multiplicative functional and effecting the scaling limit transition for the corresponding determinantal probability measures. The detailed proof of Theorem 4.1 will be published in the sequel to this note.

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