Scales of Functions and Applications in Dynamical Systems.

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Outline

Hardy'a class $\mathcal{E}\mathcal{L}$ (of exponential-logarithmic functions)

G. H. Hardy in his book "Orders of Infinity" introduced a class $\mathcal{E}\mathcal{L}$ of real functions (exponential-logarithmic functions) defined in a neighborhood of $+\infty$ by means of certain formula involving

1. the variable x,
2. the real constants,
3. algebraic (field) operations,
4. the functional symbols $\exp(\cdots)$ and $\log(\cdots)$.

EXAMPLE and REMARKABLE FACT

Example:

$$f(x) = \pi + \exp\left(7 + \frac{5 - log^3(x + log(x))}{x - log(x^2 - 5))}\right) \in \mathcal{EL}.$$

$\mathcal{EL}$-functions are germs at infinity. (Interested in the behavior at $+\infty$.)

REMARKABLE FACT

Theorem (G. H. Hardy)

Hardy class $\mathcal{EL}$ (of continuous germs at ∞) is linearly ordered by the relation of eventual dominance (at $+\infty$).

In other words, $\mathcal{EL}$ is a *scale* (see the next slide).

Definition of scales (of functions)

We denote by B the set of real continuous germs at $+\infty$.

B forms a ring under the pointwise algebraic operations.

The set B is partialy ordered by the relation $\gg$ of eventual dominance at ∞: $\boxed{f \gg g \text{ means } f(x) > g(x) \text{ for all large } x}$.

Definition

A subset $S \subset B$ is called a scale if it is linearly ordered by the relation $\gg$:

$$\boxed{f, g \in S \Rightarrow (\text{either } f \gg g, \text{ or } f \ll g, \text{ or } f = g)}.$$

By G. Hardy's theorem (previous slide), $\mathcal{EL}$ is a scale.

Scales. Related terminology

Denote

1. $B_+ = \{f \in B \mid f \gg \mathbf{0}\}$ – positive functions (germs)
2. $B_- = \{f \in B \mid f \ll \mathbf{0}\}$ – negative functions
3. $B_\sim = B \setminus \left(B_+ \cup B_- \cup \{\mathbf{0}\}\right)$ – oscillating functions

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Thus $\boxed{B = B_+ \cup B_- \cup B_\sim \cup \{\mathbf{0}\}}$ is a partition of B.

Definition

$f, g \in B$ are called comparable (notation $f \lll\!\!\Longleftrightarrow\!\!\ggg g$) if $f - g \notin B_\sim$ (i. e. if either $f \ll g$, or $f = g$, or $f \gg g$).

Thus, $S \subset B$ is a scale if $\boxed{f, g \in S \implies f \lll\!\!\Longleftrightarrow\!\!\ggg g}$

(any two functions are comparable).

Examples of scales. Questions.

EXAMPLES (of scales).

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{R}[x] \subset \mathbb{R}(x) \subset \mathbb{R}(x, e^x) \subset \mathcal{EL} \subset \ldots$$

More: Real closures of $\mathbb{Q}(x)$, $\mathbb{R}(x)$, the ring $\mathbb{R}[[x]]$ of analytic functions at ∞ with the real coefficients etc.

QUESTION. How to construct large scales of functions? In particular, extensions of e. g. $\mathcal{EL}$.

More Questions.

QUESTIONS. How much the class $\mathcal{EL}$ can be extended without losing the comparability property?

- What if we add the integrals?

- The Euler's Γ-function?

- The Riemann ζ-function?

- Solutions of various (differential, difference, functional) equations?

New method to construct scales

We describe a METHOD to construct scales of functions.

It is based on certain procedure and starts by fixing certain property (properties) P a scale may satisfy.

Properties of a scale.

A scale may satisfy certain properties P:

- $(+,-,\times)$ - being closed under listed arithmetical operations,
- D - being closed under D (the differentiation),
- $\mathcal{R}$ or $\mathcal{F}$ - to form a ring or field under pointwise operations, $\mathcal{R} = (+,-,\times)$.
- $\mathcal{H} = (\mathcal{F}, D)$ - being a Hardy field.
- ∇_1 (or ∇) - being closed under integer (or real) translations.

Special property: Hardy field. P$=\mathcal{H}$.

Definition

A scale is called a P-scale if it satisfies property P.

Definition

A scale is called an $\mathcal{H}$-scale, or Hardy field, if it satisfies property $\mathcal{H} = (\mathcal{F}, D)$, i. e. if it forms a differential field.

For example, $\mathbb{R}(x, e^x)$, $\mathbb{R}[[x]], \ldots$. Non-trivial example: Hardy's class $\mathcal{EL}$.

The property $\mathcal{H}$ is very important – mostly for historical reasons. Other properties (weaker, or stronger, or incomparable).

Excellent and Good functions.

Let P be a *reasonable* property of a scale. By Zorn's lemma,

Every P-scale is contained in a maximal P-scale.

Introduce the following classes of functions:

$E(P)$ – the class of P-excellent functions defined as:

the intersection of all maximal P-scales;

$G(P)$ – the class of P-good functions defined as:

the union of all P-scales.

$$E'(P) = \{f \in B \mid f \Lleftarrow\Rrightarrow g, \text{ for all } g \in G(P)\}.$$

(P-comparable functions, see next page).

P-comparable, excellent and good function.

Again:

$$E'(P) = \{f \in B \mid f \Longleftrightarrow\!\!\!\!\!\!\!\!\ll\gg g, \quad \text{for all} \;\; g \in G(P)\} = \\ = \{f \in B \mid f - g \notin B_{\sim}, \;\text{for all} \;\; g \in G(P)\}$$

Thus $E'(P)$ – the class of P-comparable functions – contains continuous germs (the functions in B) which are comparable with all P-good functions (functions in $G(P)$).

The following inclusions are almost immediate.

1. $E(P) \subset G(P)$ (excellent functions are good)

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2. $E(P) \subset E'(P)$, (P-excellent functions are P-comparable):
$$\big(g \in G(P),\, f \in E(P)\big) \implies f \Longleftrightarrow\!\!\!\!\!\!\!\!\ll\gg g.$$

Surprising phenomenon.

Some simple properties P lead to large scales $E(P)$ of excellent functions.

Even some very simple properties like $P = D$, or $P = (D, +)$, or $P = (\nabla, +)$.

My thesis (1981) was about the class $E(\mathcal{H})$, i. e. $P = \mathcal{H}$.

Thus $E = E(\mathcal{H})$ is the intersection of all maximal Hardy fields.

How large is it? A priori it contains $\mathbb{Q}$ – the rational constants.

It turns out that the class is much larger than that.
(See next page).

The class E=E($\mathcal{H}$) is large!

1. $\mathcal{EL} \subset E$. (E contains Hardy's class $\mathcal{EL}$).

2. E is closed under the integration (see next slide)

3. More generally, E contains many solutions of differential equations. Details.

4. E is closed under the composition.
   (If it is defined. See the slide after the next).

   More precisely: $E \circ E^+ \subset E$ where

$$E^+ = \{f \in E \mid \lim_{x\to\infty} f(x) = +\infty\}.$$

   Thus E^+ forms a semigroup under the composition operation.

The class E=E($\mathcal{H}$) is close under integration $\int$

It is enough to show that every maximal $\mathcal{H}$-scale K is closed under integration $\int$. Attempt of the proof:

Assume to the contrary that $f' = g \in K$ for some $f \notin K$.

We claim that then $f \lll\!\!\Longrightarrow\!\!\ggg k$, $\forall k \in K$. (f is comparable with every $k \in K$).

If not, then $f - k \in B_\sim$ (is oscillating) and hence also

$$(f-k)' = g - k' \in K \cap B_\sim,$$

a contradiction because $K \cap B_\sim = \emptyset$.

Thus $K \cup \{f\}$ is a scale while K was assumed to be maximal. (This is not a full proof of the claim in the title of this frame.)

My $ 100 QUESTION (34 years old)

QUESTION. Is E^+ closed under the compositional inversion?

We claim that if $f \in E^+$ then $f^{-1} \in G \cap E'$ and it must be DA (Differentialy Algebraic).

In many special cases we can prove that $f^{-1} \in E$.

(Khovanski's class of one-dimensional functions.)

The class E=E($\mathcal{H}$) is not so large.

1. All functions in E must satisfy a non-trivial ADE (algebraic differential equation).

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2. In particular, the Euler's Γ-function lies in $G \setminus E$ (M. Rosenlicht and MB).

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3. Functions in E are eventually real analytic.

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4. The class E does not contain transexponential or trans-logarithmic functions (approaching infinity faster than all the iterates of e^x or slower than all the iterates of $\log x$).

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5. With appropriate choices of properties P, the classes $E(P)$ becomes much larger.

Which functions are good (lie in $G(P)$)?

QUESTION. Which functions are good (for $P = \mathcal{H}$)?

ANSWER. These functions $f \in B$ whose differential polynomial $p(f, f', f'', \dots)$ never oscillate.

Remark: Good functions don't need to be C^∞. Explanation.

Illuminating Example

$$\exp_5(x) + \sin x \notin G \quad \text{because} \quad \exp_5(x) \in E.$$

FACT. There are good transexponential and translogarithmic functions (1986 MB).

APPLICATION: UD MOD 1.

Theorem (MB, 1994)

Let $f \in G(\mathcal{H})$ *(be good) and assume that* $f(x)$ *is subpolynomial (does not grow faster than polynomials):*

$$-x^n \ll f(x) \ll x^n, \quad \text{for some } n \in \mathbb{N}.$$

Then:

1. $(f(n))_1^\infty$ *is dense* (mod 1) *if and only if*
$$\lim_{x\to\infty} |f(x) - p(x)| = \infty, \quad \text{for every } p \in \mathbb{Q}[x].$$

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2. $(f(n))_1^\infty$ *is u.d.* (mod 1) *if and only if*
$$\lim_{x\to\infty} \frac{|f(x) - p(x)|}{\log x} = \infty, \quad \text{for every } p \in \mathbb{Q}[x].$$

Also, there is a precise conditions for well distribution etc.

Several remarks.

1. The result generalizes for “good“ sequences of regular growth other than produced by Hardy fields (e.g., produced by a large class of recurrent relations).

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2. The density result is in fact a u.d. result if appropriately interpreted (some Banach limit).

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3. Multidimensional generalization.

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4. Application for convergence in L^2 of ergodic averages along subsequences associated with Hardy fields.

Other applications.

1. For sequences good for pointwise ergodic theorem (along those sequences), [MW, MB] and [M.Wierdl, A.Quas, G.Kolesnik, MB].
2. A version of Szemeredi Thm. [M. Wierdl, N. Frantzikinakis]
3. Poincare recurrence along such sequences [MB, 1994].
4. **Waring's problem** (T. Chah, A.Kumchev and M. Wierdl 2010).
5. Nice averaging (summation) methods and Banach limits associated with Hardy fields

Examples: Properties P and classes $E(P)$

1. $P = +$ or $P = -$. Then $E = \{\mathbf{0}\}$, and G is the set of non-oscillating functions.

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2. $P = D = \frac{d}{dx}$. Then $E = \left\{c_1 e^x + c_2 e^{-x} + p(x)\right\}$ where $p(x) \in \mathbb{R}[x]$ are polynomials and $c_i \in \mathbb{R}$ are constants.

   Prove in detail that: $e^x \in E$ but $e^{2x} \notin E$.

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3. $P = (D, \times\mathbb{R}, +)$. Then $E = \left\{ \sum p_i(x) \cdot \exp(c_i x) \right\}$,

   where $c_i \in \mathbb{R}$ and $p_i(x) \in \mathbb{R}[x]$.

P-equations

For a large class of properties P, one can show that P-excellent functions $f \in E(P)$ must satisfy a non-trivial P-equation – an equation in terms of the operations in the description of the property P.

Thus for $P = D$, the only possible P-equations are

$$f^{(m)} = f^{(n)},$$

with some $m, n \geq 0$, $m \neq n$.

For $P = \mathcal{H}$, these will be ADEs (algebraic differential equations). This is a way to see that Euler's $\Gamma(x)$ is not excellent (although it is good).

The case $P = (D, +)$.

Then $E = \left\{ \sum p_i(x) \cdot \exp(c_i x) \right\}$, where $p_i \in \mathbb{R}[x]$ and $c_i \in \mathcal{A}$ where $\mathcal{A}$ is a certain subset of algebraic numbers.

In particular, $\sqrt{2} \in \mathcal{A}$ while $\sqrt[3]{2} \notin \mathcal{A}$.

(Prove that $\exp(cx) \notin E$ for transcendental c).

Analysis by P-equations.

The case $P = (\nabla, \mathcal{F})$.

Then $E = E(\nabla, \mathcal{F})$ contains $\Gamma(x)$, e^x, e^{e^x} but not $e^{e^{e^x}}$ or functions growing faster.

Functions in E must satisfy P-equations: in this case those are algebraic difference equations.

In particular, $\ln(x) \notin E$ because $\ln(x) = \log(x)$ does not satisfy P-equations. **Proof**: analytic extension.

For $\alpha \in \mathbb{R}$, we have $x^\alpha \in E \iff \alpha \in \mathbb{Q}$.

Not known whether functions in $E = E(\nabla, \mathcal{F})$ must be differentiable. Comment.

The case $P = (D, \nabla, \mathcal{F})$.

Then $E = E(D, \nabla, \mathcal{F})$ contains both $E = E(\nabla, \mathcal{F})$ and $E = E(D, \mathcal{F})$.

P-equations: in this case those are

algebraic difference-differential equations.

There are transexponential examples of P-good functions satisfying P-equations. Not known whether $E(P)$ contains transexponential functions.

Discrete Orders of Infinity

1. Scales of sequences. Similar constructions depending on properties P.

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2. Case $P = (\mathcal{F}, \nabla_1)$.

Invariant Banach limits (ultrafilters)

Denote by $B^\infty \subset B$ bounded continuous germs.

Denote $B_0 = \{f \in B \mid \lim_{x\to\infty} f(x) = 0\}$.

Denote $$B' = B^\infty / B_0.$$

Denote by $\mathcal{M}$ the semigroup of summation methods. Those are the maps $\phi : B' \to B'$ such that

(1) $\phi(f) \gg \mathbf{0}$, if $f \gg \mathbf{0}$ (i.e., ϕ is positive)

(2) $\phi(r) = r$, for $r \in \mathbb{R} \subset B'$.

Invariant Banach limits (continued)

A summation method $\phi \in \mathcal{M}$ is called a mean if $\phi(B') = \mathbb{R}$.

If $\lim_{x\to\infty} g(x) = +\infty$, define $T_g \in \mathcal{M}$ by $T_g(f) = f(g(x))$.

Denote by $\boxed{G^+(\mathcal{H}) = \{g \in G(\mathcal{H}) \mid \lim_{x\to\infty} g(x) = +\infty\}}$

the set of functions in Hardy fields approaching $+\infty$.

Theorem

There exists a mean ϕ such that, for every $g \in G^+(\mathcal{H})$ which is neither translogarithmic nor transexponential, we have $\phi = \phi \circ T_g$, i.e. $\boxed{\phi(f(x)) = \phi(f(g(x))), \forall f \in B^\infty.}$

Superrandom and Extrarandom sequences

Definition

A bounded sequence $\mathbf{c} = (c_k)$ in $\mathbb{C}$ is called extrarandom if for every minimal uniquely ergodic transformation $T: X \to X$ of a compact metric space X and every continuous function $g: X \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n\, g(T^k(x)) = 0, \quad \text{for all } \ x \in X. \qquad \text{(c)}$$

A bounded sequence is called super-random if for every deterministic system $T: X \to X$

(a system with topological entropy 0)

and every continuous function $g: X \to \mathbb{C}$ the relation (c) holds.

Super-random and Extra-random sequences II

In other words:

the *extra-random* sequences should not correlate with the values of continuous functions over the orbits of uniquely ergodic systems, while

the *super-random* sequences should not correlate with the values of continuous functions over the orbits of deterministic systems.

Let $(X_k(w))_{k=0}^{\infty}$ be an i. i. d. sequence of random variables each taking the values in the set $\{-1, 1\}$ with equal probability $1/2$. Then the sequence $c_k = X_k(w)$ is almost sure superrandom but not extrarandom. (Positive entropy systems are never disjoint in Furstenberg's sense; and, on the other hand, there are minimal uniquely ergodic systems of positive entropy).

Super-random and Extra-random sequences IIb

Let $(X_k(w))_{k=0}^{\infty}$ be an i. i. d. sequence of random variables each taking the values in the set $\{-1, 1\}$ with equal probability $1/2$. Then the sequence $c_k = X_k(w)$ is almost sure superrandom but not extrarandom. (Positive entropy systems are never disjoint in Furstenberg's sense); and, on the other hand, there are minimal uniquely ergodic systems of positive entropy).

Thus superrandom sequences do not need to be extrarandom. We pose the following question (next page).

Superrandom and Extrarandom sequences III

Thus superrandom sequences do not need to be extrarandom. We pose the following question.

Question A. Is every extrarandom sequence superrandom?

The question is motivated by my recent discovery of the non-trivial (and some colleagues find it surprising) fact of existence of extrarandom sequences.

Theorem

For any non-integral $\alpha > 0$, *the sequence* $(e^{2\pi i\, n^\alpha})_{n=1}^\infty$ *is extrarandom. The sequence* $(e^{2\pi i\,(n^2+\sqrt{n})})$ *is extrarandom while, for any real polynomial* $P(x)$, *the sequences* $(e^{2\pi i\, P(n)})$ *and* $(e^{2\pi i\,(P(n)+\log n)})$ *are not.*

Superrandom and Extrarandom sequences 4

In fact, we have a complete characterization of *subpolynomial** functions g lying in Hardy fields for which $(e^{2\pi i\, g(n)})_{n=1}^{\infty}$ is extrarandom.

subpolynomial=growing not faster than polynomials*

In our terminology, P. Sarnak's celebrated conjecture (see [**?**]) claims that the sequence $\boldsymbol{\mu} = (\mu(n))_{n=1}^{\infty}$ (of Möbius function values) is superrandom.

Question B. Is $\boldsymbol{\mu}$ extrarandom?

Related theorem

Theorem

Let X be a compact metric space and let $T : X \to X$ be a minimal uniquely ergodic transformation, let $f : X \to \mathbb{R}$ be a continuous function.

Then, for every non-integer $\alpha > 0$ and every $x \in X$, the sequence

$$f(T^n(x)) + n^\alpha$$

is u. d. $\pmod 1$.

Remark. The sequence $u(n) = n^\alpha$ in the above theorem can be replaced by any real sequence $v(n)$ such that

1. $exp(2\pi i\, v(n))$ is extrarandom;
2. $v(n)$ is u. d. $\pmod 1$