DYNAMICAL AND ALGEBRAIC PROPERTIES OF ALGEBRAIC ACTIONS

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Actions of countable discrete groups Γ on compact (metrizable) groups X by (continuous) automorphisms are a rich class of dynamical systems, and have drawn much attention since the beginning of ergodic theory. The fact that $\mathbb{Z}[\mathbb{Z}^d]$ is a commutative factorial Noetherian ring plays a vital role for such study, as it makes the machinery of commutative algebra available.

Recently, via operator algebra method, especially group von Neumann algebras, much progress has been made towards understanding the algebraic actions of general countable groups. In this report I will describe recent results joint with Hanfeng Li [3] and Andreas Thom [4] about dichotomies between dynamical property of algebraic actions and algebraic property of the corresponding modules over group rings.

Let Γ be a countable group. An action of Γ on a compact abelian group by automorphisms is called an *algebraic action*. For a locally compact abelian group X, we denote by $\widehat{X}$ its Pontryagin dual. Then we have the following correspondences:

$\{$countable left $\mathbb{Z}\Gamma$-modules $\widehat{X}\} \leftrightarrow \{$ Γ-actions on discrete abelian groups $\widehat{X}\}$, and

$\{$ Γ-actions on discrete abelian groups $\widehat{X}\} \leftrightarrow \{$ Γ-actions on compact abelian groups $X\}$.

Examples:

1) For each $k \in \mathbb{N}$, we may identify the Pontryagin dual $\widehat{(\mathbb{Z}\Gamma)^k}$ of $(\mathbb{Z}\Gamma)^k$ with $((\mathbb{R}/\mathbb{Z})^k)^\Gamma = ((\mathbb{R}/\mathbb{Z})^\Gamma)^k$ naturally. Under this identification, the canonical action of Γ on $\widehat{(\mathbb{Z}\Gamma)^k}$ is just the left shift action on $((\mathbb{R}/\mathbb{Z})^k)^\Gamma$. If J is a left $\mathbb{Z}\Gamma$-submodule of $(\mathbb{Z}\Gamma)^k$, then $\widehat{(\mathbb{Z}\Gamma)^k/J}$ is identified with

$$\{(x_1, \dots, x_k) \in ((\mathbb{R}/\mathbb{Z})^\Gamma)^k : x_1 g_1^* + \cdots + x_k g_k^* = 0, \text{ for all } (g_1, \dots, g_k) \in J\}.$$

2) Principal algebraic action: Let $f \in \mathbb{Z}\Gamma$. Put $X_f := \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f} = \{g \in (\mathbb{R}/\mathbb{Z})^\Gamma : f \cdot g = 0\}$. Let α_f be the right shift action, i.e $\alpha_{f,\gamma}(g) = g \cdot \gamma^{-1}, \forall \gamma \in \Gamma, g \in \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}$. Then $\alpha_f : \Gamma \curvearrowright X_f = \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}$ is called a *principal algebraic action*.

An action of Γ on a compact space X is called *expansive* if there is a constant $c > 0$ such that $\sup_{s\in\Gamma} \rho(sx, sy) > c$ for all distinct x, y in X, where ρ is a compatible metric on X. The definition does not depend on the choice of ρ.

Previously, characterizations of expansiveness for algebraic actions have been obtained in various special cases, such as the case $\Gamma = \mathbb{Z}^d$ for $d \in \mathbb{N}$ [15], the case Γ is

Date: July 22, 2014.

abelian [14], the case $\widehat{X} = \mathbb{Z}\Gamma/J$ for a finitely generated left ideal J of $\mathbb{Z}\Gamma$ [7], the case X is connected and finite-dimensional [1], and the case $\widehat{X} = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ for some $f \in \mathbb{Z}\Gamma$ [6].

Lemma:[3] Let $k \in \mathbb{N}$, and $A \in M_k(\mathbb{Z}\Gamma)$ be invertible in $M_k(\ell^1(\Gamma))$. Denote by J the left $\mathbb{Z}\Gamma$-submodule of $(\mathbb{Z}\Gamma)^k$ generated by the rows of A. Then the canonical action α of Γ on $X_A := \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$ is expansive.

The canonical actions α of Γ on $X_A = \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$ as above are the largest expansive algebraic actions in the sense that every expansive algebraic action is the restriction of one of these actions to a closed invariant subgroup.

Theorem:[3] Let Γ act on a compact abelian group X by automorphisms. Then the following are equivalent:
(1) the action is expansive;
(2) there exist some $k \in \mathbb{N}$, some left $\mathbb{Z}\Gamma$-submodule J of $(\mathbb{Z}\Gamma)^k$, and some $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$ such that the left $\mathbb{Z}\Gamma$-module $\widehat{X}$ is isomorphic to $(\mathbb{Z}\Gamma)^k/J$ and the rows of A are contained in J;
(3) the left $\mathbb{Z}\Gamma$-module $\widehat{X}$ is finitely generated, and $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} \widehat{X} = \{0\}$.

Theorem:[3] Suppose that Γ is amenable. Let α be an action of Γ on a compact abelian group X by automorphisms and $f \in \mathbb{Z}\Gamma$. Then f is a non-zero divisor of $\mathbb{Z}\Gamma$ if and only if $h(\alpha_f)$ is finite, where $h(\alpha_f)$ is the topological entropy of α_f .

A point $x \in X$ is called *homoclinic* if the pair (x, e_X) is asymptotic, i.e. $sx \to e_X$ when $\Gamma \ni s \to \infty$. Then the set $\Delta(X)$ of homoclinic points of X is a subgroup of X. The following result is an extension of main results in [12].

Theorem: [3] Let Γ be a polycyclic-by-finite group. Let Γ act on a compact abelian group X expansively by automorphisms. Then the following hold:

(1) The action has positive entropy if and only if $\Delta(X)$ is nontrivial.
(2) The action has completely positive entropy with respect to the normalized Haar measure of X if and only if $\Delta(X)$ is dense in X.

The left group von Neumann algebra $\mathfrak{L}\Gamma$, is defined as the closure of $\mathbb{C}\Gamma$ under the strong operator topology. Explicitly, $\mathfrak{L}$ consists of $T \in B(\ell^2(\Gamma))$ commuting with the right regular representation of Γ on $\ell^2(\Gamma)$, i.e., $(T(h\gamma))_{\gamma'} = (Th)_{\gamma'\gamma}$ for all $h \in \ell^2(\Gamma)$ and $\gamma, \gamma' \in \Gamma$, where $(h\gamma)_{\gamma''} = h_{\gamma''\gamma}$ for all $\gamma, \gamma'' \in \Gamma$. The algebra $\mathfrak{L}\Gamma$ has a canonical tracial state $\mathrm{tr}_{\mathfrak{L}\Gamma}$ defined as $\mathrm{tr}_{\mathfrak{L}\Gamma}(T) = \langle Te_\Gamma, e_\Gamma \rangle$.
The *Fuglede-Kadison determinant* for an invertible $u \in \mathfrak{L}\Gamma$ is defined as

$$\det\nolimits_{\mathrm{FK}}(u) = \exp(\mathrm{tr}_{\mathfrak{L}\Gamma} \log |u|) = \exp(\frac{1}{2}\mathrm{tr}_{\mathfrak{L}\Gamma} \log(u^* u)),$$

where $|u| = (u * u)^{1/2}$ is the absolute part of u.
When $f \in \mathfrak{L}\Gamma$ and is not invertible, we define $\det_{\mathrm{FK}}(f) = \lim_{\varepsilon \to 0} \det_{\mathrm{FK}}(|\mathrm{f}| + \varepsilon)$.
Examples: 1) For any $n \in \mathbb{N}$ and any invertible $u \in B(\ell^2_n)$, one has $\det_{FK}(u) = |\det u|^{1/n}$.

2) In the case $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$: Let $f \in \mathbb{Z}\Gamma$. Using Fourier transform we can calculate trace of f by integral on d-tori: $\mathrm{tr}(f) = \int_{\mathbb{T}^d} f(s)ds$. Then for any f is invertible in A, $\det_{\mathrm{FK}}(f) = \mathbb{M}(f)$, where $\mathbb{M}(f) = \exp(\int_{\mathbb{T}^d} \log|f(s)|\, ds)$ is the *Mahler measure* of f.

Theorem:[11] Let Γ be an amenable group and $f \in \mathbb{Z}\Gamma$. If f is a non-zero divisor of $\mathbb{Z}\Gamma$ then $h(\alpha_f) = \log \det_{\mathrm{FK}}(f)$.

The relation between entropy of principal algebraic action and Fuglede-Kadison determinant was established before for special cases such as: when $\Gamma = \mathbb{Z}^d$ [13]; when f is invertible in $\ell^1(\Gamma)$ and Γ is virtually nilpotent [5]; and for general amenable groups and f is invertible in $\mathfrak{L}\Gamma$ [10].
Recently, entropies for actions of sofic groups have been introduced by Lewis Bowen [2]; David Kerr and Hanfeng Li [9]; and Ben Hayes extended the above Theorem to algebraic actions of sofic groups [8].
Let $\mathcal{M}$ be a countable left $\mathbb{Z}\Gamma$-module. We define $\rho(\mathcal{M}) := h(\Gamma \curvearrowright \widehat{\mathcal{M}}) \in [0, \infty]$, where $h(\Gamma \curvearrowright \widehat{\mathcal{M}})$ is the topological entropy. The quantity $\rho(\mathcal{M})$ is called *torsion* of $\mathcal{M}$. Put $\mu_p := \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. Note that there is an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mu_p \to 0,$$

which induces an exact sequence

$$0 \to \mathrm{Tor}(\mu_p, \mathcal{M}) \to \mathcal{M} \to \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \mathcal{M} \to \mu_p \otimes_{\mathbb{Z}} \mathcal{M} \to 0,$$

for any abelian group $\mathcal{M}$. Here, we have $\mathrm{Tor}(\mu_p, \mathcal{M}) = \{x \in \mathcal{M} \mid \exists k \in \mathbb{N}\ p^k x = 0\}$. If $\mathcal{M}$ is a $\mathbb{Z}\Gamma$-module satisfying $\rho(\mathcal{M}) < \infty$, we set

$$\rho_p(\mathcal{M}) := \rho(\mathrm{Tor}(\mu_p, \mathcal{M})) - \rho(\mu_p \otimes_{\mathbb{Z}} \mathcal{M}).$$

In analogy to the finite places, we set $\rho_\infty(\mathcal{M}) = \rho(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{M})$ for any $\mathbb{Z}\Gamma$-module $\mathcal{M}$.
Theorem:[4] Let $\mathcal{M}$ be a $\mathbb{Z}\Gamma$-module with finite torsion. Then, we have

$$\rho(\mathcal{M}) = \rho_\infty(\mathcal{M}) + \sum_p \rho_p(\mathcal{M}).$$

Moreover, for any exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ of $\mathbb{Z}\Gamma$-modules with finite torsion, we have $\rho_p(\mathcal{M}) = \rho_p(\mathcal{M}') + \rho_p(\mathcal{M}'')$ for any prime p, and $\rho_\infty(\mathcal{M}) = \rho_\infty(\mathcal{M}') + \rho_\infty(\mathcal{M}'')$.

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