Questions in Dynamics and Numbers

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Outline

We pose miscellaneous questions (in Dynamics and Numbers) and discuss briefly the underlying and related material.

Most questions can be subdivided into the following four subjects:

1. various recurrence theorems
2. IETs and billiards in polygons (flat surfaces)
3. APs (arithmetical progressions)
4. others

Poincare Pointwise Recurrence Theorem (PPRT)

SETTING:

Let X be a probability measure space (X, Φ, μ) endowed with a compatible metric d so that (X, d) is separable.

We also assume that $T : X \to X$ is measure preserving $(\mu(T^{-1}(A)) = \mu(A),\ \forall A \in \Phi)$.

Theorem (Pointwise Poincare Recurrence Theorem)

Under the above assumptions on the system (X, T), we have

$$\liminf_{k \to \infty} d(x, T^k(x)) = 0, \quad \text{for } \mu\text{-a.a. } x \in X.$$

In the next slide we state a “moving” version of the above theorem.

"Moving" version of PPRT

(=Pointwise Poincare Recurrence Theorem)

Conjecture. Let n_k be an arbitrary sequence of non-negative integers. Under the above assumptions on (X, T),

$$\liminf_{k\to\infty} d(T^{n_k}x, T^{k+n_k}(x)) = 0, \quad \text{for } \mu\text{-a.a. } x \in X.$$

Remarks:

1. The claim of Conjecture is easily validated for bounded $\{n_k\}$.
2. The conjecture survived serious attack (special semester at MSRI, Fall 2008), it is validated in several special cases.
3. A topological version of the above conjecture has been validated in a joint paper with E. Glasner (2009). (See next slide).

Topological version of moving recurrence Thm

Theorem (Joint with E. Glasner)

Let $T : X \to X$ be a minimal continuous transformation of a compact metric space (X, d). Let $\{n_k\}$ be an arbitrary sequence of non-negative integers. Then, for a residual set of $x \in X$,

$$\liminf_{k\to\infty} d(T^{n_k}x, T^{k+n_k}(x)) = 0. \tag{3}$$

Even under the conditions of the above theorem, we don't know whether the set of recurrent points needs to have full measure.

On the other hand, we prove that the power T^{k+n_k} can be replaced (in (3)) by $T^{r_k+n_k}$, for arbitrary recurrent sequence r_k (e.g. $T^{k^2+n_k}$ with $r_k = k^2$).

Recurrence in the unit interval

It has been proved (MB, 1993) that PPRT can be quantified under some minimal assumptions on the space (X, d). It suffices for (X, d) to be σ-compact, or to have a finite Hausdorff dimension.

In particular, the following result is obtained as a corollary of a more general result (exercise).

Theorem (Recurrence in the unit interval)

Let $X = [0, 1]$ and assume that $T : X \to X$ be a Lebesgue measure preserving map. Then for a. a. $x \in X$ we have

$$\liminf_{n\to\infty} |T^n(x) - x| \le c = 1.$$

We pose the question on the best constant c in the above theorem. We know that $\boxed{\frac{1}{\sqrt{5}} \le c \le \frac{1}{2}}$. We conjecture that $\boxed{c = \frac{1}{\sqrt{5}}}$.

Bibliography

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I. Shkredov.

One-dimensional recurrence of sums

The following one-dimensional recurrence result is well known.

Theorem

Let (X, T) *be ergodic and measure preserving where* (X, Φ, μ) *is a probability measure space. Let* $f \in L^1(X, \Phi, \mu)$ *be real valued and such that* $\int f \, d\mu = 0$. *Denote*

$$S_n(x) = \sum_{k=1}^{n} f(T^k(x)), \quad n \geq 0.$$

Then the sequence $S_n(x)$ *is almost sure recurrent to zero:*

$$\liminf_{n\to\infty} |S_n(x)| = 0, \qquad \text{for a. a. } x \in X.$$

I am interested in the following generalization (next slide).

Exotic one-dimensional recurrence

We assume that T_1, T_2 are both ergodic and measure preserving on the same measure space (X, Φ, μ) and that $f_1, f_2 \in L^1(X, \Phi, \mu)$ are real valued and such that

$$\int f_1 \, d\mu = \int f_2 \, d\mu.$$

Denote

$$S'_n(x) = \sum_{k=1}^{n} \Big(f_1(T_1^k(x)) - f_2(T_2^k(x)) \Big), \quad n \geq 0.$$

We conjecture: $S'_n(x)$ almost sure recurs to 0.

If it helps, one may assume that T_1, T_2 commute or even that $T_2 = T_1^2$ (one of the T's is a power of another one).

A conjecture of similar nature - next slide.

Special case

We assume that T is μ-preserving on a measure space (X, Φ, μ) and that $f \in L^1(X, \Phi, \mu)$ is real valued. Denote

$$\boxed{S_n(x) = \sum_{k=0}^{n-1} f(T^k(x))} \quad \text{and} \quad \boxed{S_n^{(2)}(x) = S_{2n}(x) - 2S_n(x)}.$$

We conjecture that $S_n^{(2)}(x)$ almost sure recurs to 0.

Note that if f is a measurable coboundary plus a constant (even without the assumption $f \in L^1$), then the above conjecture is validated by using Roth theorem.

Combinatorial application: Next page.

Combinatorial Application: PolyMath Project

Let $x = \{x_k\}$ be a bounded sequences of integers.

We conjecture: There are TWO consecutive blocks in x of equal lengths and equal sums. That means: $\boxed{\sum_{j=n}^{n+k-1} x_j = \sum_{j=n+k}^{n+2k-1} x_j}$, for some $n, k \in \mathbb{N}$. (Moreover, both n, k can be selected arbitrary large).

Note, that if only "of equal sums" is required then it follows from van der Waerden Theorem (in fact, for ANY number of blocks).

It has been shown only recently that TWO in the above conjecture cannot be replaced with THREE

(ArXiV 2014: Julien Cassaigne, James D. Currie, Luke Schaeffer, Jeffrey Shallit)

The above conjecture has been presented for the polymath project by Terence Tao in 2009 (google: "boshernitzan problem").

Complexity of ϵ-skipping directions

Consider billiards on a rational polygon Γ.

It is well known (since Katok, Zemlyakov ≈1975) that "non-dense" directions for orbits form a countable dense subset of S^1.

Denote by $D_\epsilon(\Gamma)$ the set of ϵ-skipping directions. Those are directions which contain an infinite trajectory which fails to be ϵ-dense on the table Γ. It is easy to see that $D_\epsilon(\Gamma) \subset S^1$ must be countable closed subsets.

If Γ is a "tiling" polygon (a rectangle, some triangles, etc.) then $D_\epsilon(\Gamma)$ must be finite. But already for some "almost integrable" polygons (e.g., which are skew products over the irrational rotations) the set $D_\epsilon(\Gamma)$ does not need to be finite.

QUESTION. What are the possible complexities of the sets $D_\epsilon(\Gamma)$?

Complexity of ITMs and Billiards

ITMs - interval translation maps (Non-bijective version of IETs).The study has been initiated in a joint paper with I. Kornfeld in 1995. It is easily seen that the complexity of ITMs is at most polynomial.

Conjecture. Complexity of non-periodic ITMs is linear (just like for IETs). (Hubert Pascal, Buzzy Jerome, ...).

Irrational billiards. Generally believed that it is at most polynomial, Katok proved that it is sub exponential (and hence the entropy is 0). Shcheglov received explicit sub exponential estimates (far from polynomial).

Does there exist a minimal polygonal billiard?

A polygon Γ will be called minimal (uniquely ergodic) if every infinite orbit in it is dense (uniformly distributed) in the phase space of the corresponding billard dynamical systems. (Such Γ must be irrational.)

It is known (Katok, Zemlyakov) that "typical" polygonal billiards (residual subset) possess a dense orbit. In fact, "most" orbits are uniformly distributed (approx. argument in Kerckhoff, Masur, Smillie 1986).

One can show (also by an approximation argument) that the set of non-dense directions in a typical billiard table forms an arbitrary small set (in the metric sense: can be shown to have Hausd. dim. 0).

Conjecture(s). **There are no minimal (uniquely ergodic) polygonal billiards.**

Note: Such billiards cannot contain periodic orbits.

Healthy vector spaces for IETs.

For an IET (interval exchange transformation) T, denote by $V(T) \subset \mathbb{R}$ the vector subspace over $\mathbb{Q}$ generated by the lengths of the intervals exchanged by T. We define $\mathrm{rank}(T) = \dim_{\mathbb{Q}} V(T)$.

Definition

A vector space $V \subset \mathbb{R}$ (over $\mathbb{Q}$) is called *strange* if there exists an IET T with $V(T) \subset V$ which is minimal but not uniquely ergodic. Otherwise V is called healthy.

Theorem (MB, 1988)

If $\mathrm{rank}(T) = 2$, then minimality implies unique ergodicity.
Reformulation*: $V \subset \mathbb{R}$ is healthy if $\dim(V) \leq 2$.*

Conjecture. For every $k \geq 3$, "most" vector subspaces $V \subset \mathbb{R}$ of dimension k are healthy. (State a restatement).

Problem. Show that there are healthy vector subspaces $V \subset \mathbb{R}$ of any finite dimension $d \geq 3$.

Genericity of weak mixing for induced maps

Setting: Let $T : X \to X$ be (Lebesgue) measure preserving and ergodic transformation of the unit interval $X = [0, 1)$.

Denote by $T_c : [0, c) \to [0, c)$ the induced maps, $0 < c < 1$. Finally, let $\boxed{NWM(T) = \{c \in (0, 1) \mid T_c \text{ is not weakly mixing}\}}$.

Conjecture. NWM is a small set.

The conjecture is validated for some low complexity systems, in particular for all (ergodic) IETs ([MB, 2012]).

Question. Do there exist persistently weakly mixing IETs?

APs in orbits of IETs

Notation: (1) IET(s)=interval exchange transformation(s)
(2) AP - arithmetical progression
(3) RAP - rich in APs = contains arbitrary long non-constant APs

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Definition

An IET $T : X \to X$ is called RAP if it contains a RAP orbit. T is strongly RAP if $\forall N \geq 1$, $\exists n > N, k \geq 1, x \in X$ such that $\boxed{x, T^k x, T^{2k} x, \ldots, T^{(n-1)k} x}$ forms an AP.

If T is RAP and minimal then each its orbit is RAP.

For examples see next slide.

APs in orbits of IETs, continued

The following minimal IETs are RAP:

1. Lebesgue a.a., also a residual set (assuming the permutation is irreducible)
2. Rank 2 IETs (RAP but maybe not strongly RAP, van der Waerden)
3. minimal non-uniquely ergodic 4-IETs

We don't have examples of minimal IETs which are not RAP. Nevertheless, we conjecture the following.

Conjecture. Pseudo Anosov (and maybe all linearly recurrent) IETs not reducing to rotations fail to be RAP (e.g., Rauzy's example with SAF=0).

AAPs - approximate APs

Definition

Let (X,d) be a metric space and $n \geq 3$. Then X is said to satisfy AAP_n (contain Almost Arithmetic Progressions of length n) if for every $\epsilon > 0$ there are n distinct points $(x_k)_{k=1}^n$ such that $\left|\frac{x_i - x_j}{x_2 - x_1} - |i-j|\right| < \epsilon$.

(X,d) is said to satisfy AAP if it is AAP_n for all $n \geq 3$.

One can show that the AAP property of a metric space is a bilipschitz invariant. One can also show that the subsets in $[0,1)$ of full Hausdorff dimension must be AAP, while there exists a compact set $K \subset [0;1)$ of Hausdorff dimension $.99$ which is not even AAP_3.

AAPs - approximate APs in plane

Conjecture A. Any connected subset $X \subset \mathbb{R}^2$ containing more than one point is AAP_3.

We have recently validated the above conjecture for X being path connected (strong version).

Conjecture B. The graph G_f of any function $f : [0, 1] \to [0, 1]$ is AAP_3. (For continuous f - easy calculus problem).

Not true if AAP_4 (rather than AAP_3), already for continuous f.

Superrandom and Extrarandom sequences

Definition

A bounded sequence $\mathbf{c} = (c_k)$ in $\mathbb{C}$ is called extra-random if for every minimal uniquely ergodic transformation $T: X \to X$ of a compact metric space X and every continuous function $g: X \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n\, g(T^k(x)) = 0, \quad \text{for all } \; x \in X. \qquad \text{(c)}$$

A bounded sequence is called super-random if for every deterministic system $T: X \to X$

(a system with topological entropy 0)

and every continuous function $g: X \to \mathbb{C}$ the relation (c) holds.

Super-random and Extra-random sequences II

In other words:

the *extra-random* sequences should not correlate with the values of continuous functions over the orbits of uniquely ergodic systems, while

the *super-random* sequences should not correlate with the values of continuous functions over the orbits of deterministic systems.

Let $(X_k(w))_{k=0}^{\infty}$ be an i. i. d. sequence of random variables each taking the values in the set $\{-1, 1\}$ with equal probability $1/2$. Then the sequence $c_k = X_k(w)$ is almost sure superrandom but not extrarandom. (Positive entropy systems are never disjoint in Furstenberg's sense; and, on the other hand, there are minimal uniquely ergodic systems of positive entropy).

Super-random and Extra-random sequences IIb

Let $(X_k(w))_{k=0}^{\infty}$ be an i. i. d. sequence of random variables each taking the values in the set $\{-1, 1\}$ with equal probability $1/2$. Then the sequence $c_k = X_k(w)$ is almost sure superrandom but not extrarandom. (Positive entropy systems are never disjoint in Furstenberg's sense); and, on the other hand, there are minimal uniquely ergodic systems of positive entropy).

Thus superrandom sequences do not need to be extrarandom. We pose the following question (next page).

Superrandom and Extrarandom sequences III

Thus superrandom sequences do not need to be extrarandom. We pose the following question.

Question A. Is every extrarandom sequence superrandom?

The question is motivated by my recent discovery of the non-trivial (and some colleagues find it surprising) fact of existence of extrarandom sequences.

Theorem

For any non-integral $\alpha > 0$, *the sequence* $(e^{2\pi i\, n^\alpha})_{n=1}^\infty$ *is extrarandom. The sequence* $(e^{2\pi i\,(n^2+\sqrt{n})})$ *is extrarandom while, for any real polynomial* $P(x)$, *the sequences* $(e^{2\pi i\, P(n)})$ *and* $(e^{2\pi i\,(P(n)+\log n)})$ *are not.*

Superrandom and Extrarandom sequences 4

In fact, we have a complete characterization of *subpolynomial** functions g lying in Hardy fields for which $(e^{2\pi i\, g(n)})_{n=1}^{\infty}$ is extrarandom.

subpolynomial=growing not faster than polynomials*

In our terminology, P. Sarnak's celebrated conjecture (see [**?**]) claims that the sequence $\boldsymbol{\mu} = (\mu(n))_{n=1}^{\infty}$ (of Möbius function values) is superrandom.

Question B. Is μ extrarandom?

Related theorem

Theorem

Let X be a compact metric space and let $T: X \to X$ be a minimal uniquely ergodic transformation, let $f: X \to \mathbb{R}$ be a continuous function.

Then, for every non-integer $\alpha > 0$ and every $x \in X$, the sequence

$$f(T^n(x)) + n^\alpha$$

is u. d. $\pmod 1$.

Remark. The sequence $u(n) = n^\alpha$ in the above theorem can be replaced by any real sequence $v(n)$ such that

1. $exp(2\pi i\, v(n))$ is extrarandom;
2. $v(n)$ is u. d. $\pmod 1$

Algorithm for deciding on the minimality of IET.

Let $T : X \to X$ be an IET, $X = [0, 1)$ over $V = V(T)$. Denote by $U_n(T)$ the following finite subset:

$$U_n(T) = \{T^n x - x \mid x \in X\} \subset V.$$

A set U in a vector space V is called balanced if its closed convex span contains $\mathbf{0} \in V$. (Otherwise U is called unbalanced).

Question. If $T : X \to X$ is a minimal IET with the SAF invariant (flux) not vanishing, does it meen that $U_n(T)$ is out of balance for some n (and then for all large n) ?

The answer is affirmative under the assumption that T is uniquely ergodic.

The question is relevant to establishing an algorithm for validating minimality of a given IET T. (Done in detail for rank 2 IETs).