Higher Rank Orbit Closures in $\mathcal{H}^{\mathrm{odd}}(4)$ and Beyond

1 Introduction

It is a long standing problem to understand the $\mathrm{SL}(2,\mathbb{R})$ orbit closures of translation surfaces in moduli space. Most recently, a significant breakthrough was made in the work of Eskin and Mirzakhani, and Eskin, Mirzakhani, and Mohammadi, [4, 5] where they showed that every orbit closure is an affine invariant manifold.

Before the theorems of [4, 5], there was already progress in genus two due to McMullen [6] and independently Calta [3]. In $\mathcal{H}(2)$ either an orbit is closed, i.e. it is a Teichmüller curve, or it is dense in the stratum. In $\mathcal{H}(1,1)$, they showed that there are intermediate dimensional orbit closures given by Prym eigenforms. Otherwise, the orbit is again either a Teichmüller curve or dense in the stratum.

In $\mathcal{H}(4)$ there are two connected components: $\mathcal{H}^{\mathrm{hyp}}(4)$ and $\mathcal{H}^{\mathrm{odd}}(4)$. It was shown by Nguyen and Wright [7], that all orbits in $\mathcal{H}^{\mathrm{hyp}}(4)$ are either Teichmüller curves or dense in $\mathcal{H}^{\mathrm{hyp}}(4)$.

In joint work with Duc-Manh Nguyen and Alex Wright, we prove that the Prym locus is the unique intermediate dimensional orbit closure in $\mathcal{H}^{\mathrm{odd}}(4)$. Recall that the Prym locus $\tilde{\mathcal{Q}}(3,-1^3)$ is the canonical double covering of the stratum of quadratic differentials on the torus with a triple zero and three simple poles.

Theorem 1.1 ([1] Thm. 1.1). *The only proper higher rank affine invariant submanifold of $\mathcal{H}^{odd}(4)$ is the Prym locus $\tilde{\mathcal{Q}}(3,-1^3)$.*

This theorem and its complete proof can be found in [1].

2 Background

Let (X,ω) be a translation surface written as a pairing of a Riemann surface carrying an Abelian differential. Let Σ denote the finite set of singularities of ω on X. Let $\{\gamma_1,\ldots,\gamma_n\} \subset H_1(X,\Sigma,\mathbb{Z})$ be a basis for relative homology. Period coordinates are defined by the map

$$\Phi : (X,\omega) \mapsto \left(\int_{\gamma_i} \omega\right) \in \mathbb{C}^n.$$

By [4, 5], period coordinates provide local coordinate charts for the orbit closure $\mathcal{M}$ of (X, ω).

We consider the tangent space to an affine manifold in a subspace of relative cohomology, i.e. $T_{\mathbb{C}}(\mathcal{M}) \subset H^1(X, \Sigma, \mathbb{C})$. We can restrict to the real tangent space corresponding to $T_{\mathbb{R}}(\mathcal{M}) \subset H^1(X, \Sigma, \mathbb{R})$. Then there is a canonical projection

$$p : H^1(X, \Sigma, \mathbb{R}) \to H^1(X, \mathbb{R}),$$

which can be restricted to the tangent space to yield $p(T_{\mathbb{R}}(\mathcal{M})) \subset H^1(X, \mathbb{R})$. It was proven in [2], that $p(T_{\mathbb{R}}(\mathcal{M}))$ is symplectic so, in particular, it has even dimension. We define the rank of an orbit closure following [8], to be half the dimension of $p(T_{\mathbb{R}}(\mathcal{M}))$.

In the stratum $\mathcal{H}(4)$, p is an isomorphism. Hence, the dimension of $T_{\mathbb{R}}(\mathcal{M})$ is either 2, 4, or 6. If $p(T_{\mathbb{R}}(\mathcal{M}))$ has rank one, then the orbit is a Teichmüller curve, and if it has rank three, then the orbit is dense in the connected component. Therefore, it suffices to study orbit closures with rank two.

3 Beyond: $\mathcal{H}(n, m)$ with $n + m = 4$

This work is part of a much larger program motivated by a conjecture of Maryam Mirzakhani, known as the covering conjecture.

Conjecture (Mirzakhani)**.** *If rank$(\mathcal{M}) \geq 2$ and $k(\mathcal{M}) = \mathbb{Q}$, then $\mathcal{M}$ is either a stratum or a covering of a stratum.*

By the work of [6], this is confirmed in genus two, and by the work of [7] and [1] in the stratum $\mathcal{H}(4)$.

Joint work in progress with Duc-Manh Nguyen seems to indicate that this conjecture also holds for rank two affine manifolds in the strata in genus three with two zeros. We summarize the list of rank two affine manifolds here:

- $\tilde{\mathcal{H}}_{hyp}(2) = \tilde{\mathcal{Q}}(1, 1, -1, -1) \subset \mathcal{H}^{hyp}(2, 2)$
- $\tilde{\mathcal{H}}_{odd}(2) \subset \tilde{\mathcal{Q}}(4, -1^4) \subset \mathcal{H}^{odd}(2, 2)$
- There do not exist rank two affine manifolds in $\mathcal{H}(3, 1)$.

References

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