Bispaces and Bibundles

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Introduction

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- Joint work with David Roberts and Danny Stevenson
- I'll put the talk on my webpage
- There should be a paper on the arXiv ... soon.

Outline

Why bibundles?

- If G is a Lie group a G-bibundle is a principal (right) G-bundle $P \to M$ which has an additional free left G action commuting with the right action and having the same orbits.
- These are needed in the definition of gerbes for a (non-abelian) group G where you would like to be able to form a product of two principal G-bundles.
- This is not generally possible for principal G-bundles unless G is abelian.
- However if $P \to M$ and $Q \to M$ are bibundles you can form a product $P \otimes Q \to M$ by forming fibrewise

$$(P \otimes Q)_m = (P_m \times Q_m)/G$$

where G acts by $(p, q)g = (pg, g^{-1}q)$.
- $P \otimes Q$ is also a bibundle.

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- Bibundles are not a new idea. They certainly goes back to work of Breen on bitorsors in 1990.
- Also discussed by Aschieri, Cantini, and Jurco in 2005.
- However when you look for examples there are not as many of them as there are principal bundles.
- Our aim is to address this existence question.
- It turns out that we need to use crossed modules instead of just Lie groups G.
- While my coworkers are keen crossed module and 2-group people I resisted this at first.
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G-bispaces

- It is simplest to work at first with a typical fibre. i.e a bibundle over a point. This is a G-bispace that is a set X having commuting, left and right, free and transitive G-actions.

Examples 1

- Let G be abelian and X a right G-space. Define $g \star x \star h = xh(g^{-1})$.
- This only works when G is abelian. Otherwise left and right actions don't commute.
- We regard this bispaces as uninteresting examples.

Examples 2

- Take $X = G$ with the usual left and right G action. Call this the trivial bispace T.
- Fix $\xi \in \mathrm{Aut}(G)$ and define X with the action $g \star x \star h = \xi^{-1}(g)xh$. Call this bispace $T(\xi)$.

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- Fix $\xi \in \mathrm{Aut}(G)$ and define X with the action $g \star x \star h = \xi^{-1}(g)xh$. Call this bispace $T(\xi)$.

- The left and right G-actions are related by the structure map

$$\psi : X \to \mathrm{Aut}(G)$$

defined by $xg = \psi(x)(g)x$.
- The structure map is equivariant in the sense that $\psi(xg) = \psi(x) \circ \mathrm{Ad}(g)$.

Lemma 3 (Breen)

The construction of the structure map defines an equivalence between

1. *G-bispaces X.*
2. *Pairs (X, ψ) consisting of a right G-space X and an equivariant map $\psi : X \to \mathrm{Aut}(G)$.*

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1. *G-bispaces X.*
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The Type of a G-bispace

- There is a natural notion of a morphism of G-bispaces X and Y. This is a map $f: X \to Y$ commuting with the G-actions.
- From the equivariance of the structure map it has image in an orbit of $\mathrm{Ad}(G)$ on the right of $\mathrm{Aut}(G)$ and thus defines an element of $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Ad}(G)$. We call this the type of X and denote it $\mathrm{Type}(X)$.

Example 4 ($T(\xi)$)

The structure map is defined by $x \star h = \psi(x)(h) \star x$ and we have $g \star x \star h = \xi^{-1}(g)xh$. It follows that $xh = (\xi^{-1}(\psi(x)(h)))x$ or $\xi(xhx^{-1}) = \psi(x)(h)$ and hence

$$\psi(x) = \xi \circ \mathrm{Ad}(x) \qquad \text{and} \qquad \mathrm{Type}(T(\xi)) = [\xi]$$

where $[\xi]$ is the image under $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ of ξ.

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Denote by Bisp_G the set of all G-bispaces. We have

Lemma 5

Two G-bispaces X and Y are isomorphic if and only if

$$\mathrm{Type}(X) = \mathrm{Type}(Y)$$

As every element of $\mathrm{Out}(G)$ arises as the type of some $T(\xi)$ we have

Proposition 6

The isomorphism classes of G-bispaces are in bijective correspondence with $\mathrm{Out}(G)$ via the type map

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Properties of the type map

- If X and Y are G-bispaces we have seen how to define a new G-bispace $X \otimes Y$.
- We can also define a dual X^* to be the same set but a new action $g \star x \star h = h^{-1}xg^{-1}$.

Lemma 7

The map $\mathrm{Type}\colon \mathrm{Bisp}_G \to \mathrm{Out}(G)$ *satisfies*

1. $\mathrm{Type}(X \otimes Y) = \mathrm{Type}(X)\,\mathrm{Type}(Y)$
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- We say that the type map is multiplicative.

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Changing structure group of a G-bispace

- If X is a right G-space and $f\colon G \to K$ a homomorphism there is a natural right K-space X_K defined by $X_K = (X \times K)/G$ where the G action is $(x,k)g = (xg, f(g)^{-1}k)$ and the K-action on equivalence classes is $[x,k]k' = [x,kk']$.
- There is a map $X \to X_K$ satisfying the obvious equivalance condition relative to $f\colon G \to K$.
- What about G-bispaces? It usually doesn't work.
- The way to make it work is to choose (if you can) a homomorphism $\tilde{f}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(K)$ such that $\tilde{f} \circ \mathrm{Ad}_G = \mathrm{Ad}_K \circ f$.
- Now use the equivalence of bispaces and right spaces with structure map from Lemma 3. Given X a right G-space with structure map ψ_G then X_K is a right K-space with structure map $\psi_K([x,k]) = \tilde{f}(\psi(x))\,\mathrm{Ad}(k)$.
- This is telling us we should be using crossed modules.

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Crossed-modules

A crossed module is a generalisation of the pair G, $\mathrm{Aut}(G)$. More precisely:

Definition 8

A *crossed module* is a pair of groups (H,G) with homomorphisms

$$G \xrightarrow{t} H \xrightarrow{\alpha} \mathrm{Aut}(G)$$

such that

1. $t(\alpha(h)(g)) = ht(g)h^{-1}$ and;
2. $\alpha \circ t = \mathrm{Ad}_G$.

Note that

- (1) $\Rightarrow G_1 = \ker(t) \subset Z(G)$ the centre of G, and hence $\ker(t)$ is abelian,
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Examples of crossed modules

Example 9

The pair $(\mathrm{Aut}(G), G)$ is a crossed module

$$G \overset{\mathrm{ad}}{\to} \mathrm{Aut}(G) \overset{\mathrm{id}}{\to} \mathrm{Aut}(G)$$

Example 10

For any group G there is a crossed module

$$1 \to G \to \mathrm{Aut}(1) = 1$$

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There is a crossed module

$$A \to 1 \to \mathrm{Aut}(A)$$

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Example 12

If G is a normal subgroup of H then the adjoint action of H on H fixes G and this defines a homomorphism $\alpha\colon H \to \mathrm{Aut}(G)$. The result is a crossed module

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Example 13

In particular if PK is the group of smooth based paths $\gamma\colon [0,1] \to K$ then ΩK the group of loops ($\gamma(0) = \gamma(1) = 1$) is a normal subgroup so that we have a crossed module

$$\Omega K \to PK \to \mathrm{Aut}(\Omega K)$$

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Properties of crossed modules

There is an obvious definition of a morphism of crossed modules:

Definition 14

A morphism of crossed modules $(H, G) \to (H', G')$ consists of a pair of homomorphisms $u\colon H \to H'$ and $v\colon G \to G'$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{v} & G' \\ \downarrow t & & \downarrow t' \\ H & \xrightarrow[u]{} & H' \end{array}$$

commutes and the equivariance condition

$$v(\alpha(h)(g)) = \alpha'(u(h))(v(g))$$

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Football Theorem

Theorem 15 (Football Theorem)

Winning is not transitive.

Proof.

□

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- Ghana defeated Serbia

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(H, G)-bispaces

We have

Definition 16 (Breen)

Let (H, G) be a crossed module. An (H, G)-bispace is a pair (X, ψ) consisting of a right G-space X and an equivariant map $\psi \colon X \to H$.

- We call ψ the structure map again.
- Equivariance means $\psi(xg) = \psi(x)t(g)$ and hence defines the type of X which is now an element in $H/t(G)$. This is a group because $t(G)$ is normal.
- There is a dual and a product which are a little trickier to define. Again the type map is multiplicative.
- Again we have:

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Proposition 17

The isomorphism classes of (H, G)-bispaces are in bijective correspondence with $H/t(G)$ via the type map

$$\mathrm{Type}\colon \mathrm{Bisp}_{(H,G)} \to H/t(G)$$

(H, G)-bibundles

It is now simple to generalise to bibundles.

Definition 18

Let (H, G) be a crossed module. An (H, G)-bibundle is a (right) principal G-bundle with an equivariant map $\psi\colon P \to H$.

- Each fibre of $P \to M$ is an (H, G)-bispace.
- They may not be isomorphic as (H, G)-bispaces!
- The structure map descends to give a commuting diagram:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & H/t(G) \end{array}$$

  and we call $\phi\colon M \to H/t(G)$ the type or type map of $P \to M$.
- The value $\phi(m)$ tells you the isomorphism class of the fibre of $P \to M$ at m.
- Notice that two (H, G)-bibundles which have different type maps cannot be isomorphic.

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It is now simple to generalise to bibundles.

Definition 18

Let (H,G) be a crossed module. An (H,G)-bibundle is a (right) principal G-bundle with an equivariant map $\psi\colon P \to H$.

- Each fibre of $P \to M$ is an (H,G)-bispace.
- They may not be isomorphic as (H,G)-bispaces!
- The structure map descends to give a commuting diagram:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & H/t(G) \end{array}$$

  and we call $\phi\colon M \to H/t(G)$ the type or type map of $P \to M$.
- The value $\phi(m)$ tells you the isomorphism class of the fibre of $P \to M$ at m.
- Notice that two (H,G)-bibundles which have different type maps cannot be isomorphic.

Examples

Example 19

A G-bundle is the same thing as an $(\mathrm{Aut}(G), G)$-bibundle. The type map takes values in $\mathrm{Out}(G)$.

Example 20

If A is abelian then an A-bundle is the same thing as a $(1, A)$-bundle where we just define the structure map $\psi\colon P \to 1$ in the unique way.

Example 21

If G is normal in H then $H \to H/G$ is a G-bundle and the identity map $H \to H$ makes it an (H, G)-bibundle.

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Example 22

If $\rho\colon M \to \mathrm{Aut}(G)$ we define $T(\rho)$ by making the fibre at m the $(\mathrm{Aut}(G), G)$-bispace $T(\rho(m))$. The type map is $\phi(m) = [\rho(m)]$ the image of $\rho(m) \in Aut(G)$ in $\mathrm{Out}(G)$.

Example 23

The trivial (H, G)-bibundle over M is $P = G \times M$ with the structure map being the projection to G composed with $t\colon G \to H$.

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The trivial (H,G)-bibundle over M is $P = G \times M$ with the structure map being the projection to G composed with $t\colon G \to H$.

How much of the bibundle does the type map determine?

- Because $t(G)$ is normal in H we have that $H \to H/t(G)$ is a $(H, t(G) = G/G_1)$ bibundle.
- If we quotient P by G_1 we obtain a $G/G_1 = t(G)$-bundle. The structure map descends to $\psi\colon P/G_1 \to H$ and also defines an $(H, t(G))$-bibundle.
- These two $(H, t(G))$-bibundles are isomorphic because of

$$\begin{array}{ccc} P/G_1 & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & H/t(G) \end{array}$$

Lemma 24

If $G_1 = 1$ then $P \to M$ is the pull-back of $H \to H/t(G)$ by the type map.

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Products and duals and the type map

- We can define the product and dual of two (H,G)-bibundles fibrewise.
- If $\mathrm{Bibun}_{(H,G)}(M)$ is the set of all (H,G)-bibundles on M we let

$$\mathrm{Type}\colon \mathrm{Bibun}_{(H,G)}(M) \to \mathrm{Map}(M, H/t(G))$$

be the map sending $P \to M$ to its type map $\phi\colon M \to H/t(G)$.

Lemma 25

1. $\mathrm{Type}(P \otimes Q) = \mathrm{Type}(P)\,\mathrm{Type}(Q)$
2. $\mathrm{Type}(P^*) = (\mathrm{Type}(P)^{-1})$.

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Change of structure crossed module

- If $(H,G) \to (H',G')$ is a morphism of crossed modules applying the bispace construction pointwise gives a map

$$\mathrm{Bibun}_{(H,G)}(M) \to \mathrm{Bibun}_{(H',G')}(M).$$

which preserves products and duals.

- In particular as G_1 is abelian we have the morphism of crossed modules $(1,G_1) \to (H,G)$ defined by the obvious inclusions.
- Combining with the type map gives a sequence (of pointed sets):

$$\mathrm{Bun}_{G_1}(M) = \mathrm{Bibun}_{(1,G_1)}(M) \xrightarrow{\iota} \mathrm{Bibun}_{(H,G)}(M) \xrightarrow{\mathrm{Type}} \mathrm{Map}(M, H/t(G))$$

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If $P \to M$ is an (H,G)-bibundle then $\mathrm{Type}(P) = 1$ if and only if there is a G_1-bundle $R \to M$ such that $\iota(R) = P$.

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$$\mathrm{Bun}_{Z(G)}(M) \xrightarrow{\iota} \mathrm{Bibun}_G(M) \xrightarrow{\mathrm{Type}} \mathrm{Map}(M, \mathrm{Out}(G))$$

- In this case $\mathrm{Out}(G)$ is the group of automorphisms of the Dynkin diagram: a finite group. It follows that $\phi\colon M \to \mathrm{Out}(G)$ lifts to $\hat{\phi}\colon M \to \mathrm{Aut}(G)$.

Proposition 27

Any G-bibundle for G compact, simple, simply connected is of the form $R \otimes T(\hat{\phi})$ for R a $Z(G)$-bundle.

Moral

To get 'interesting' bibundles, i.e. those which aren't really abelian bundles in disguise, we need to use groups which have large groups of automorphisms; such as the loop group.

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Classifying theory for G-bundles

- Recall that there is a universal G-bundle $EG \to BG$, unique up to homotopy equivalence, with the property that for any G-bundle P there is a classifying map $f : M \to BG$ such that $P \simeq f^*(EG)$.
- The classifying map is unique up to homotopy.
- We want a similar result for (H, G)-bibundles.
- Notice first that if $P \to M$ is a bibundle and $f : N \to M$ then $f^*P \to N$ is a bibundle:

$$\begin{array}{ccccc} f^*P & \xrightarrow{\hat{f}} & P & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow & & \downarrow \\ N & \xrightarrow{f} & M & \xrightarrow{\phi} & H/t(G) \end{array}$$

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The bundle of bibundle structures

- The structure map $\psi\colon P \to H$ is equivalent to a section of $P \times_G H \to M$ where G acts by $(p,h)g = (pg, ht(g))$.
- In fact, given a G-bundle $P \to M$ the possible (H,G)-bibundle structures on it are the sections of $P \times_G H \to M$.
- One way to see this is to note that $P \times H \to P \times_G H$ is a G-bundle and the projection $P \times H \to H$ is a structure map making $P \times H \to P \times_G H$ into a (H,G)-bibundle.
- Any section ψ of $P \times_G H$ pulls back $P \times H$ and this is naturally identified with $P \to M$ and induces the bibundle structure defined by ψ.

$$\begin{array}{ccccc} P & \xrightarrow{(\mathrm{id},\psi)} & P \times H & \to & H \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\psi} & P \times_G H & \to & H/t(G) \end{array}$$

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The universal bibundle

- Apply the construction above to $EG \to BG$ and denote $E(H,G) = EG \times H$ and $B(H,G) = EG \times_G H$.
- This gives the universal bibundle

$$\begin{array}{ccc} E(H,G) & \stackrel{\Psi}{\to} & H \\ \downarrow & & \downarrow \\ B(H,G) & \stackrel{\Phi}{\to} & H/t(G) \end{array}$$

where Ψ is the projection from $E(H,G) = EG \times H$ onto H.

- If $P \to M$ is an (H,G)-bibundle then it has a classifying map as a G-bundle which is a pull-back diagram

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & EG \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

- The pair $\hat{F} = (\hat{f}, \psi)\colon P \to EG \times H = E(H,G)$ is G-equivariant and descends to a map $F\colon M \to B(H,G)$ giving us

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Lemma 28

In the above situation the type map ϕ of $P \to M$ satisfies $\phi = \Phi \circ F$. Conversely take any $F\colon M \to B(H,G)$ then $F^*(E(H,G)) \to M$ is an (H,G)-bibundle with type map $\Phi \circ F$.

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- Denote by $[M, B(H,G)]_\Phi$ the resulting Φ-homotopy classes.

Proposition 29

The classifying map of $P \to M$ is unique up to Φ-homotopy. Pull-back defines a bijection

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where $\mathrm{IBibun}_{(H,G)}(M)$ denotes the set of all isomorphism classes of (H,G)-bibundles.

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