MPIM topology seminar
Feb 8, 2021

3- and 4-manifolds via knots and links

Slogan: "knots and links give a concrete approach to studying 3- and 4-manifolds"

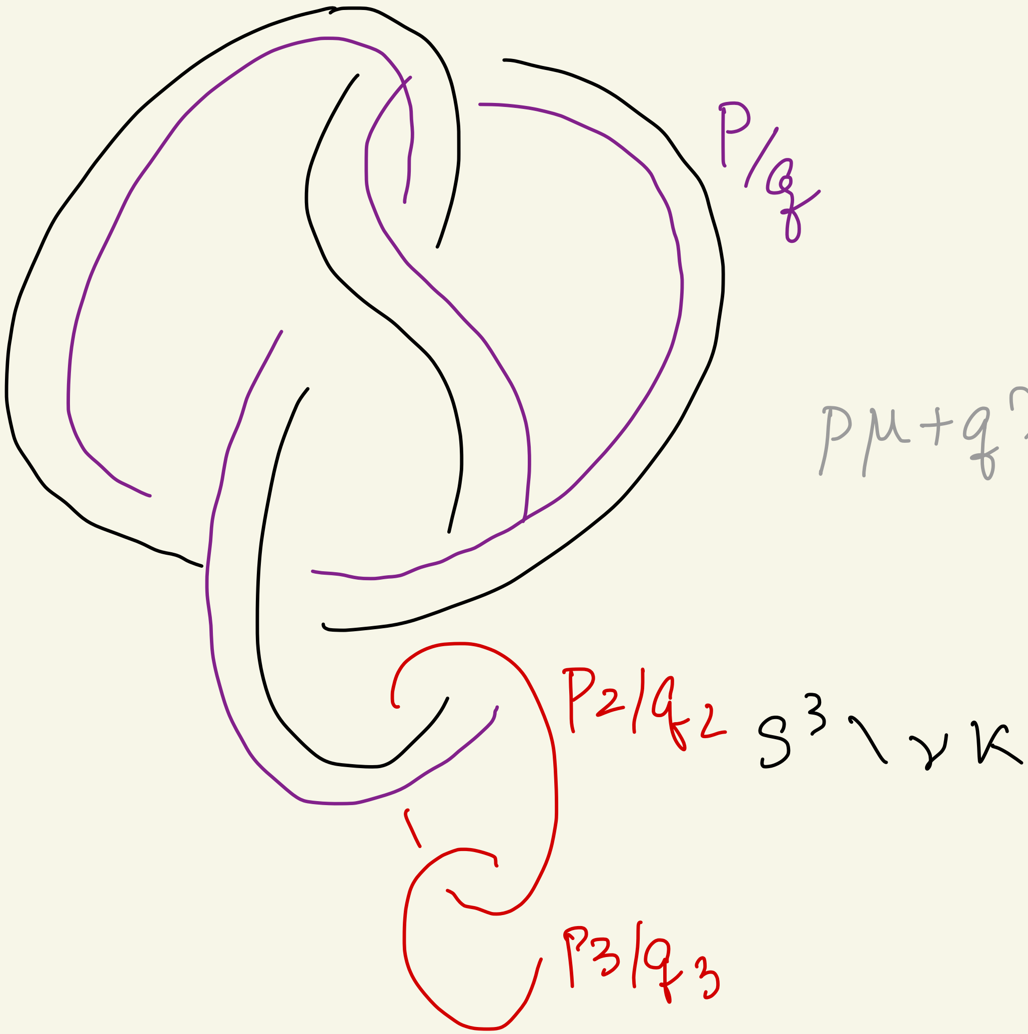
Plan:

- Dehn surgery descriptions of 3-manifolds
  - existence & uniqueness
  - 3-manifolds bound 4-manifolds [Rokhlin 1951]

- Kirby diagrams for 4-manifolds
  - examples

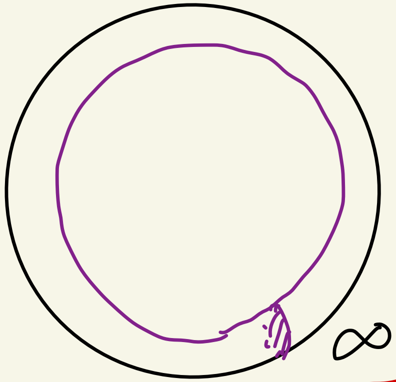
- Sample problems and conjectures
  - atomic surgery problems
  - homology cobordism group & triangulations

Dehn surgery on links



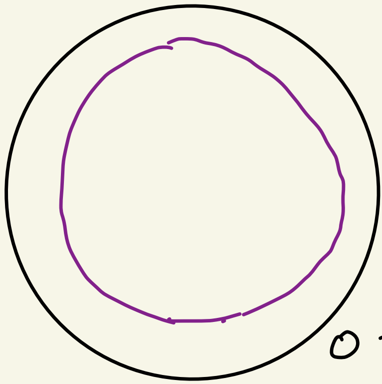
[Alex trick]: Any homeo of S^2 extends over D^3

Examples:



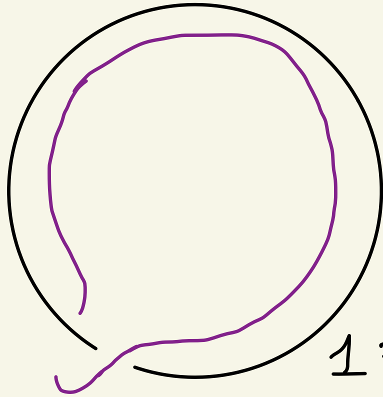
$\infty \ (= \frac{1}{0})$

$= S^3$



$0 = \frac{0}{1}$

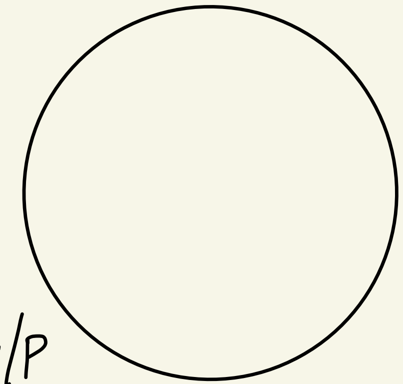
$= S^1 \times S^2$



$1 = \frac{1}{1}$

$= S^3$

$L(p,q) =$

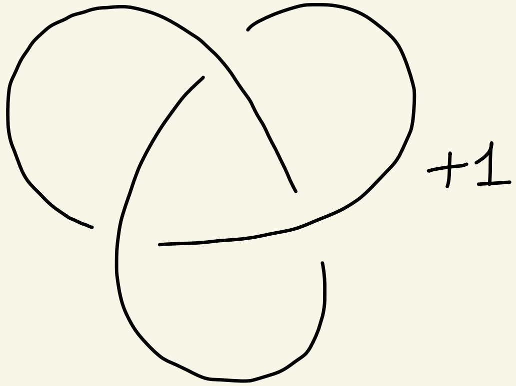


$L(1,q) \cong S^3$
$\forall q$

$\pi_1(L(p,q)) \cong \mathbb{Z}/p$

Lens spaces

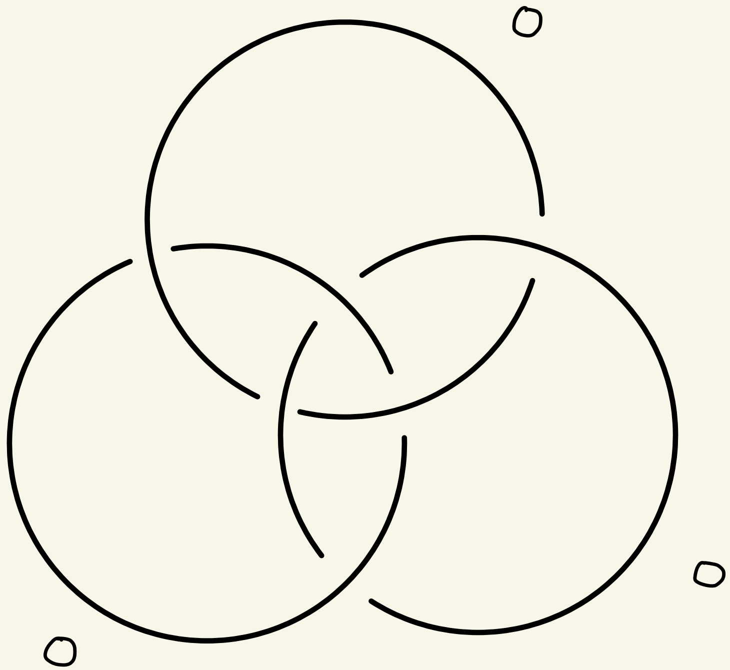
$P :=$



$H_*(P) \cong H_*(S^3)$

Poincaré homology sphere

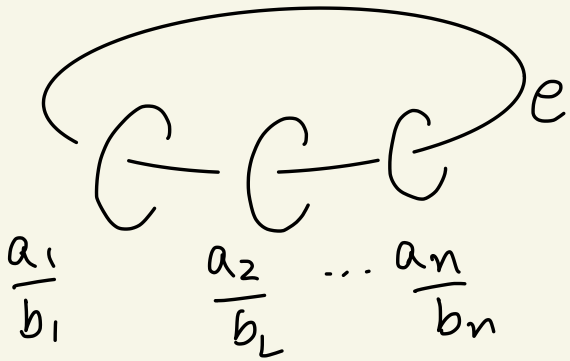
$S^1 \times S^1 \times S^1$



Seifert fibered spaces

Def: $M^3 = \bigsqcup S^1$ disjoint union of circles

equiv. $M = S^1$-bundle over an orbifold.



S^1-bundle over S^2 with euler # = e

[Lickorish-Wallace 1960] Every closed, orientable M^3 is the result of Dehn surgery on some link $L \subseteq S^3$

Sketch of proof:

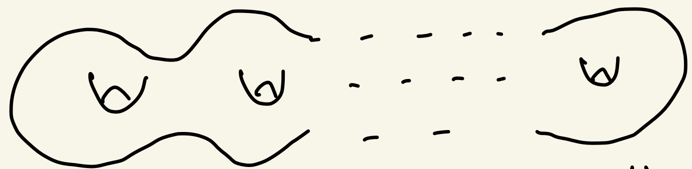
[Moise 1952] Every TOP 3-mfld admits a unique triangulation.

3-simplices

M a 3-mfld w. triangulation K.

Then $\nu(K_{(1)})$ is a handlebody

tubular nbd



H_g

So is $\nu(K^*_{(1)})$

dual triangulation

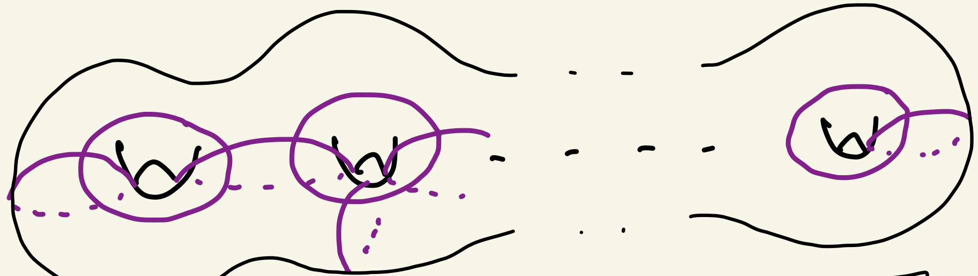
compact, connected, $\partial = \emptyset$

So, every closed, orientable M^3 admits a Heegaard splitting.

i.e. $M = H_g \cup_{\varphi : \Sigma_g \circlearrowleft \cong} H_g$

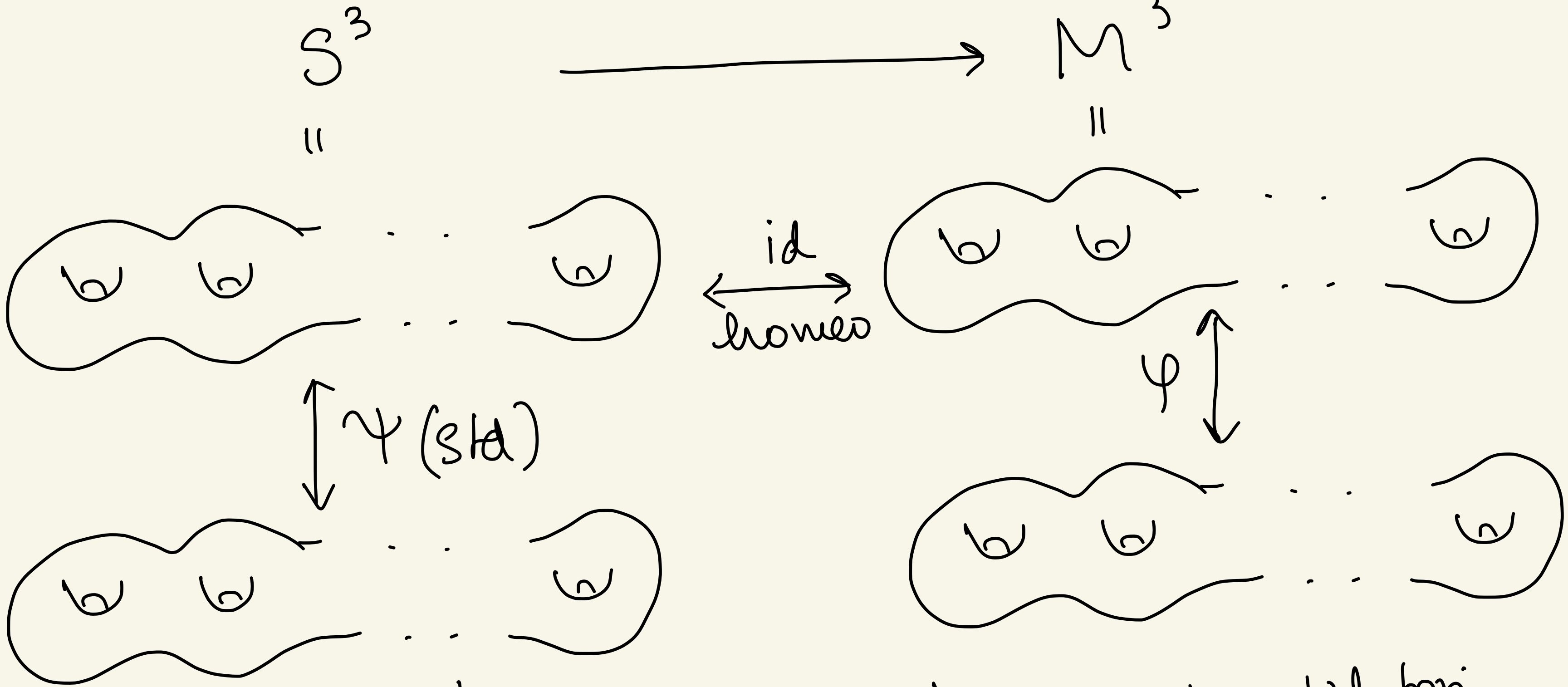
Note $S^3 = H_g \cup_{\psi : \Sigma_g \circlearrowleft} H_g$ standard.

Every $\varphi : \Sigma_g \circlearrowleft \cong$ is a composition of Dehn twists τ_i, up to isotopy

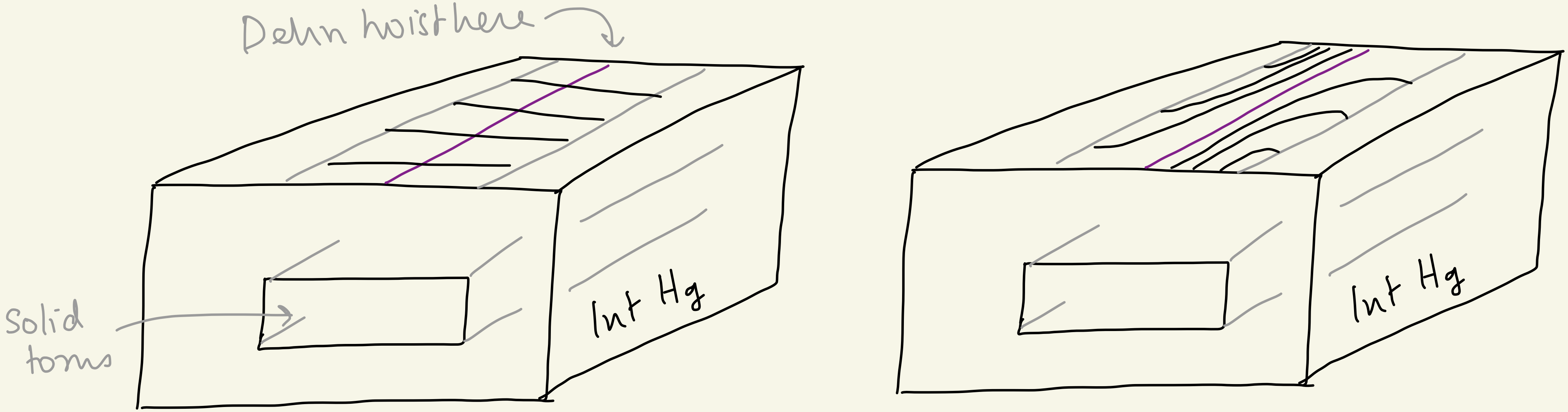


3g-1 curves

Σ_g



$\varphi^{-1}\psi$ extends to $H_g \setminus$ disjoint solid tori



Conclusion: Every closed, oriented 3-mfld is the result of ± 1-framed Dehn surgery on some link $L \subseteq S^3$

$$\frac{\{\text{framed links} \subseteq S^3\}}{\text{diagrammatic moves}} \xrightarrow{\text{Dehn surgery}} \frac{\left\{\begin{array}{c}\text{closed, orientable}\\ \text{3-manifolds}\end{array}\right\}}{\text{homeo}}$$

Lickorish-Wallace

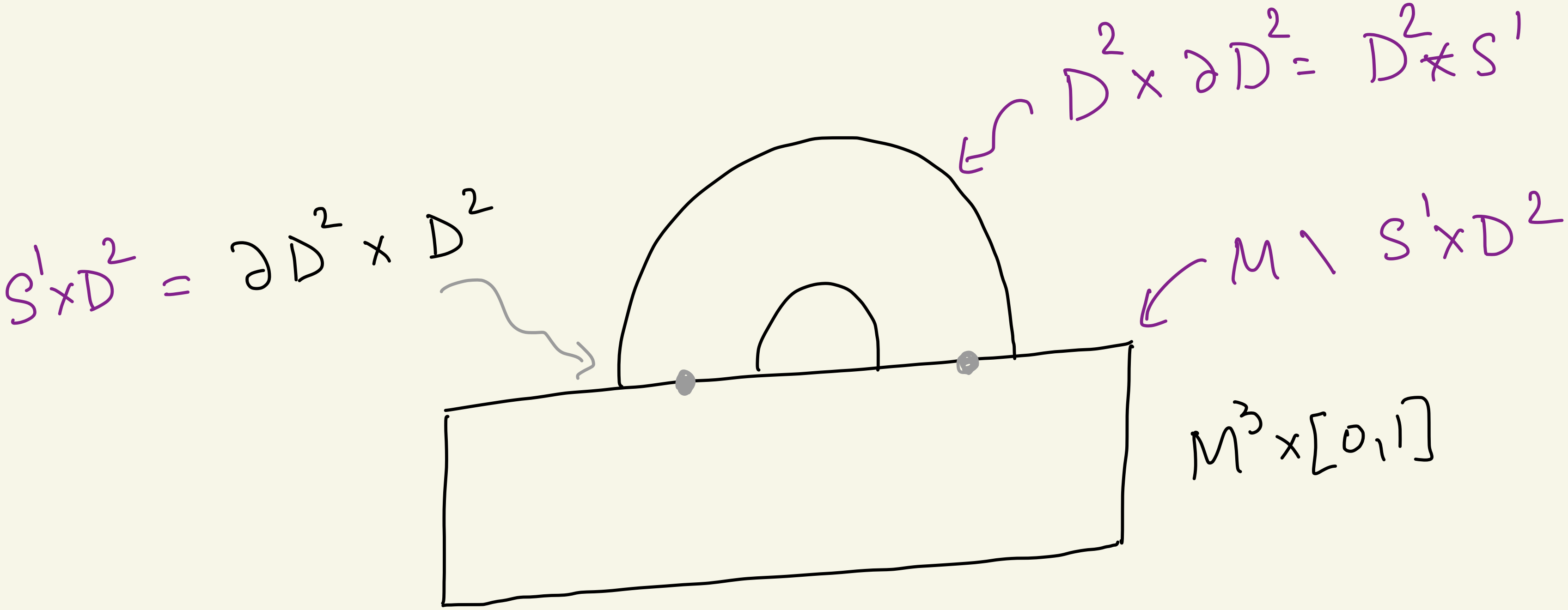
Kirby

[handle slides + blow up/down]

3-manifolds bound 4-manifolds [Rokhlin 1951]

↳ simply conn.

Geometric proof: Integral surgery $\Longleftrightarrow$ 4-dim 2-handle addition $D^2 \times D^2$



Conclusion: Every $M^3 = \partial\,(\text{simply connected } W^4)$

Questions:

Q1: $n(M^3) :=$ min $\{ n \mid M$ is the result of Dehn surgery on an n-comp link $\subseteq S^3 \}$

$n(T^3) = 3$

$\exists$ lower bounds from $rk(H_1(M))$, weight$(\pi_1(M))$ ← # normal generators

Best that we can do: $\exists M$, $rk(H_1(M)) = 1$, or 0, π_1 wt $= 1$.
$n(M) = 2$.

Conjecture: (Wiegold) Every finite pres. perfect grp has weight one ← # normal gens

Q2: $\Omega^3 = 0$

Given M^3, what kinds of 4-mflds W have $\partial W = M$?

- restrict H_*?
- require low b_2?
- aspherical?

(Smooth) 4-manifolds

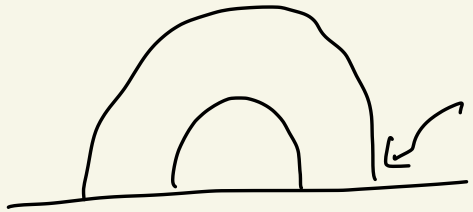
W^n smooth $\Longrightarrow$ admits a Morse function $\Longrightarrow$ admits a handle decomposition

index k critical pts $\rightsquigarrow$ n-dim index k handle

$D^k \times D^{n-k}$

attached along

$\partial D^k \times D^{n-k}$



$\partial D^k \times D^{n-k}$

Without loss of generality, assume $\exists$ single 0-handle

$\exists$ single n-handle

Dimension 4

0-handle $:= D^0 \times D^4 = D^4$

Further handles are attached on $\partial D^4 = S^3$

2 handles attached along $\partial D^2 \times D^2 = S^1 \times D^2$

$\uparrow$

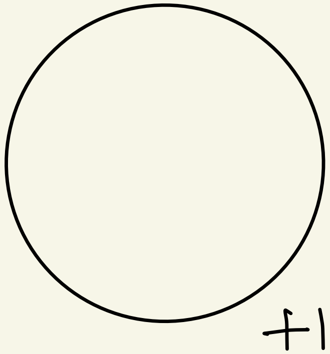
attached along (framed) knots.

[Laudenbach-Poenaru 1972] Every homeo of $\#_k S^1 \times S^2$ extends over $\natural S^1 \times D^3$

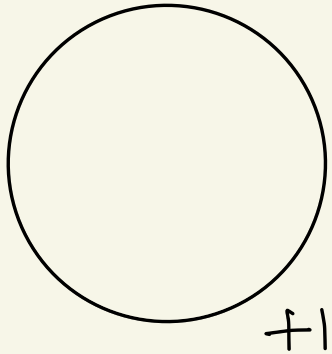
$\rightsquigarrow$ in a closed 4-manifold, $\exists$ unique way to attach the 3-h & 4h

In a closed 4-mfld, only need to explain where the 1h and 2h are attached.

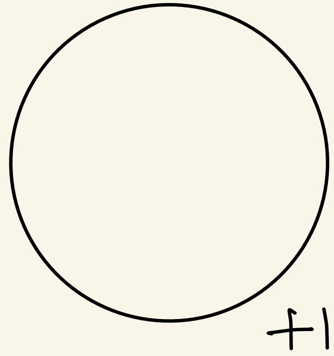
What are we looking at?



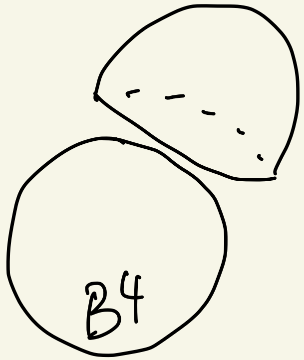
$= S^3$



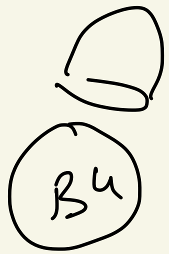
$= D^2$-bundle over S^2 with euler # $= 1$



$= CP^2$



← 2-h with framing +1

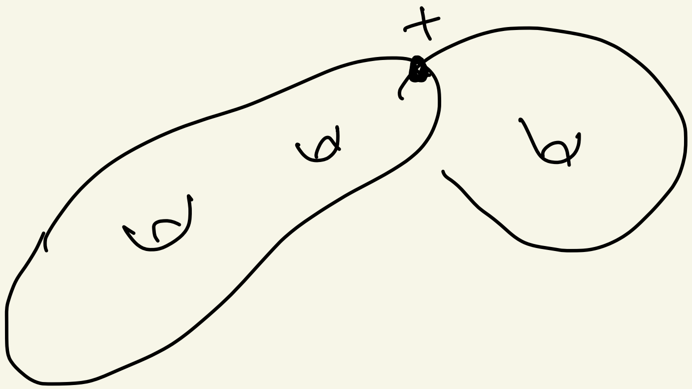


2-h

$\cup$ 4-h along S^3

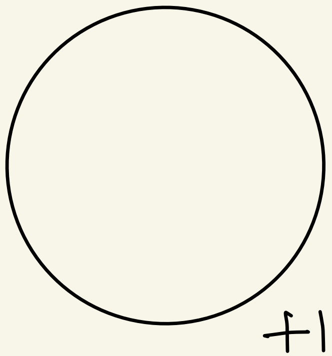
Intersection form for closed 4-mflds

$$Q_W : H_2(W^4;\mathbb{Z}) \times H_2(W;\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(x, y) \longmapsto \langle x^* \smile y^*, [W] \rangle$$

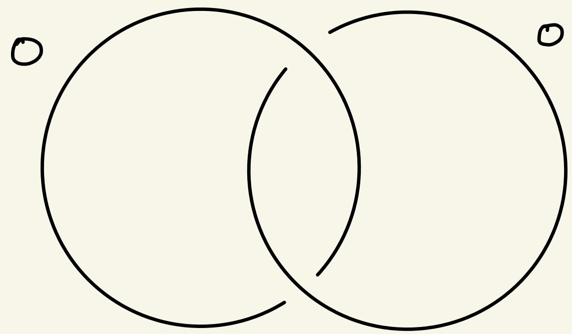


represent H_2 classes by surfaces, make transverse, then count intersection points (including sign)

e.g.

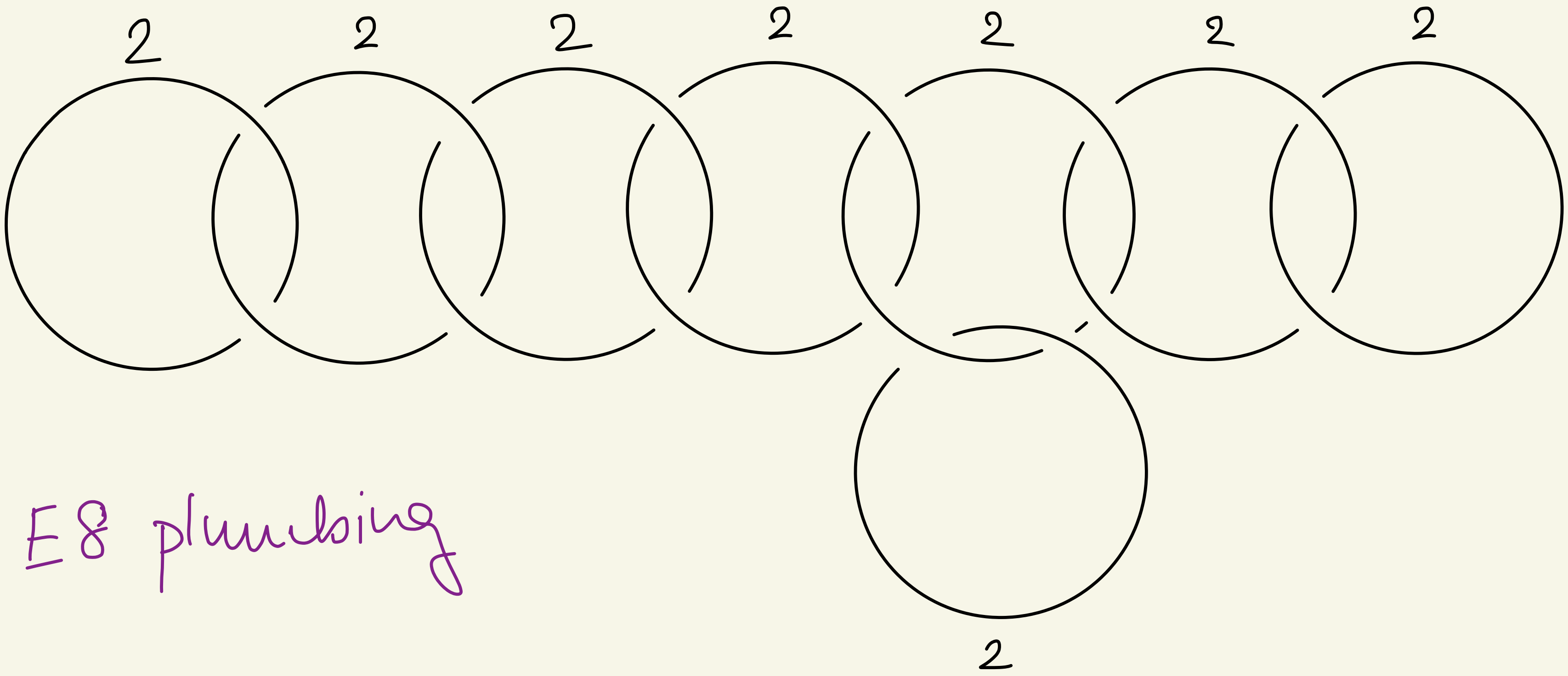


$[+1]$

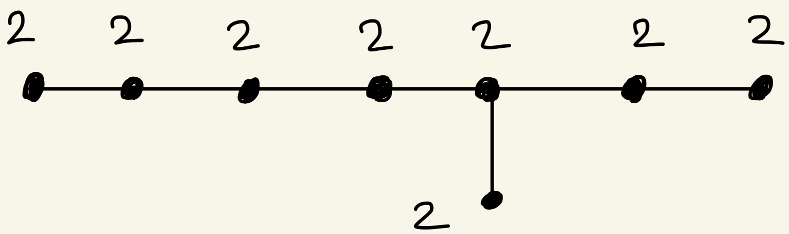


$= S^2 \times S^2$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

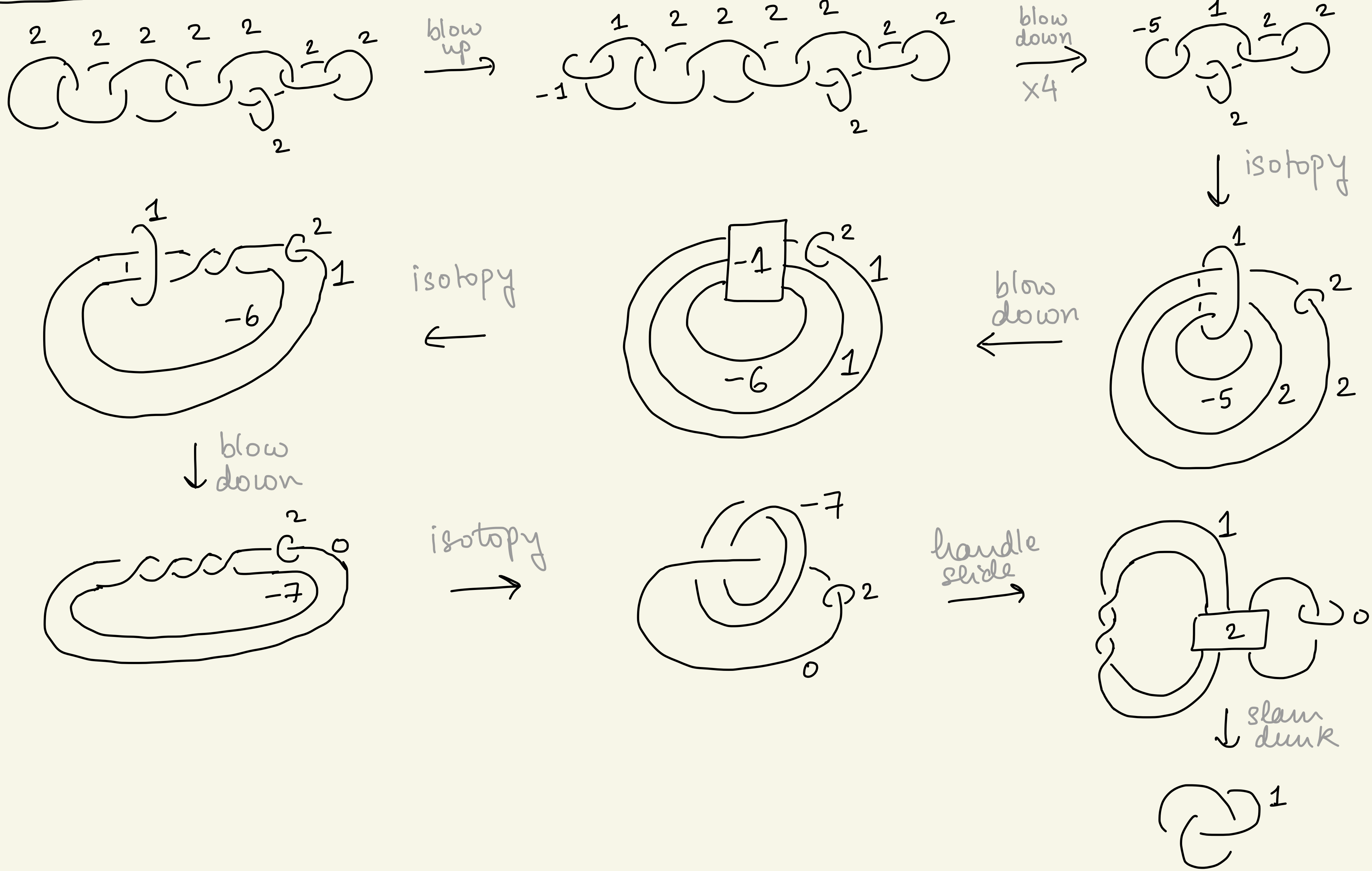


E8 plumbing



$$Q_{E8} = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & \Large 0 & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & \Large 0 & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$$

$\partial E8$ = Poincaré



Sample problems

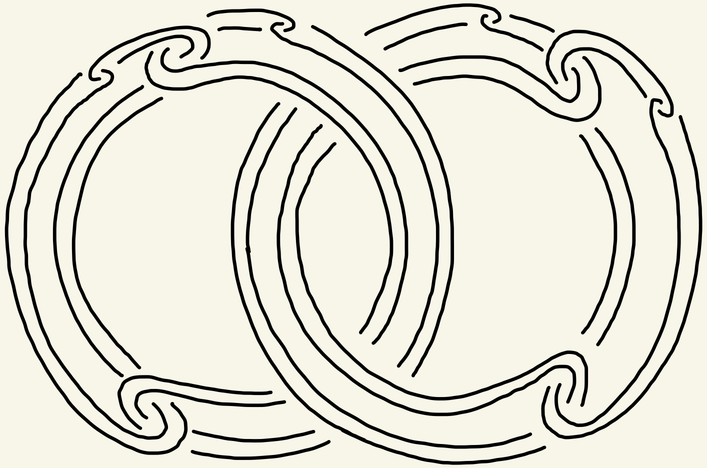
- Atomic surgery problems.

Surgery works in dim 4 in TOP category

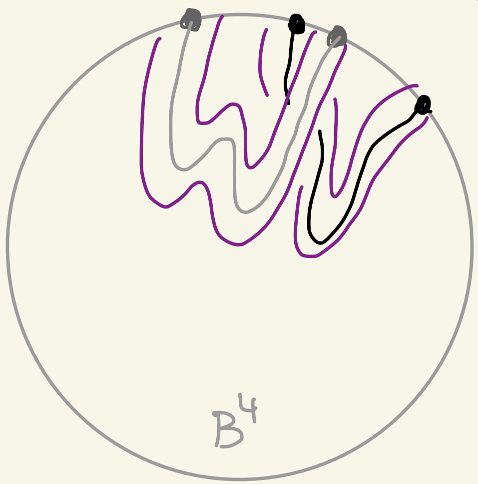
iff

every element in a certain family of links is freely slice

e.g.



Wh(Bing(Hopf))

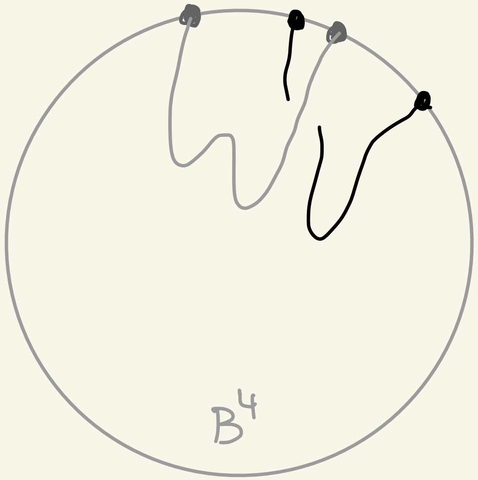


$L \subseteq S^3$ freely slice if bound disjoint discs in B^4 + $\pi_1(\text{comp}) \cong F_n$.

$L \subseteq S^3$ slice $\Longleftrightarrow$ $S^3_0(L)$ bounds a 4-mfld

(+ certain properties)

Zero surgery on each comp of L



• Triangulation conjecture: Is every M^n homeo to a simplicial complex?

A: No ($n=4$ Freedman, 1982; $n \geq 5$ Manolescu 2013)

$$\Theta_n^{\mathbb{Z}} := \left(\left\{ Y^n \text{ PL oriented} \mid H_*(Y) \cong H_*(S^n) \right\} \big/ \mathbb{Z}\text{-hom cob} \,,\ \# \right)$$

[Kervaire 1969] $\Theta_n^{\mathbb{Z}} = 0 \quad \forall n \neq 3$

Define $\mu : \Theta_3^{\mathbb{Z}} \longrightarrow \mathbb{Z}/2$ Rokhlin invariant

$Y \longmapsto \dfrac{\sigma(W)}{8}$, W compact, spin, smooth, $\partial W = Y$.

e.g. $P = \partial E8 \implies \mu(P) = 1 \implies \Theta_3^{\mathbb{Z}} \neq 0$

[Galewski-Stern, Matumoto 1980] Every M^n can be triangulated $\forall n$

iff

$\exists\, H^3 \in \Theta_3^{\mathbb{Z}}$ with $\mu(H) = 1$ and $H \# H = \partial(\text{acyclic PL 4-mfld})$

Questions ?