EHP sequence and the Goodwillie tower

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$F : Top_* \to Top_*$

$\rightsquigarrow$ tower

$$\begin{array}{c} F(X) \\ \downarrow \\ \vdots \\ \downarrow \\ D_3(F)(X) \to P_3(F)(X) \\ \downarrow \\ D_2(F)(X) \to P_2(F)(X) \\ \downarrow \\ D_1(F)(X) = P_1(F)(X) \end{array}$$

excisive approximation.

$\operatorname{holim} P_n(F)(X) \xleftarrow{\simeq} X$

"converges at X"

$\pi_* F(X)$ May not be excisive.

$\pi_* P_1(F)(X)$

homology thy in X.

$F = Id$

$\pi_*(Id(X)) = \pi_* X$ Harder.

Not excisive.

$\pi_*(P_1(Id)(X)) = \pi_*^s X$

Hard

$P_1(Id)(X) = \Omega^\infty \Sigma^\infty X$

$\mathbb{D}_n(F) : Top_* \to Sp$

$D_n(F)(X) = \Omega^\infty \mathbb{D}_n(F)(X)$

$\partial_n(F) \in Sp$ $\circlearrowleft$ free Σ_n action.

$\pi_* \mathbb{D}_n(Id)(X)$ homotopy gps of spectra.

$\mathbb{D}_n(F)(X) = \partial_n(F) \wedge_{\Sigma_n} \Sigma^\infty X^n$

$P_n(X) := P_n(Id)(X)$

$\mathbb{D}_n(X) := \mathbb{D}_n(Id)(X)$

$\partial_* = \partial_*(Id)$

Formally GSS

① Start

$E_1^{s,t} = \pi_t \mathbb{D}_s(X) \Rightarrow \pi_t(X)$ ③

answer after you take H_*

take homology w.r.t. all differentials ②

$\pi_* P_1(X) = \pi_*^s(X) = \varinjlim \pi_{n+*}(\Sigma^n X)$

Primary stability

$\pi_* P_2(X)$

Primary + secondary stability

metastable homotopy (Toda, Mahowald)

Thm (Ching) ∂_* forms an operad in spectra.

Spectral shifted Lie operad.

$\{\partial_n\}_n$

$\partial_n \in Sp$ $\circlearrowleft$ Σ_n

objects in $\mathcal{C}$ w/ this alg structure.

Σ_n

objects in $\mathcal{C}$ w/ this alg structure.

$(\mathcal{C}, \otimes)$ sym monoidal cat

$\mathcal{O}$ operad in $\mathcal{C}$ "Algebraic Structure"

$\mathcal{O}_n \in \mathcal{C}$ + operad structure maps
$\circlearrowleft$
Σ_n

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$

algebras over this operad

$$\mathcal{F}_{\mathcal{O}} : \mathcal{C} \to \mathcal{C}$$

$$\mathcal{F}_{\mathcal{O}}(X) = \bigoplus_n \mathcal{O}_n \otimes_{\Sigma_n} X^{\otimes n}$$

free $\mathcal{O}$-alg generated by X

Examples

$\mathcal{O} = \mathrm{comm}$

$$\mathcal{F}_{\mathcal{O}}(X) = \mathrm{Sym}(X) = \bigoplus_n X^{\otimes n}/\Sigma_n$$

$\mathcal{C} = (\mathrm{Ab}, \otimes)$ $\quad \mathrm{Lie}_n = \mathbb{Z}^{(n-1)!}$

$\mathcal{O}$ = Lie operad

$$\mathcal{F}_{\mathrm{Lie}}(A) = \mathrm{Lie}(A)$$

free Lie alg.

Whitehead product:

$$[-,-] : \pi_k(X) \otimes \pi_\ell(X) \to \pi_{k+\ell-1}(X)$$

$$[x,y] = (-1)^{|x||y|}[y,x]$$

Makes $\pi_*(X)$ into a shifted Lie algebra.

GSS

$$D_n(X) = \partial_n \wedge_{\Sigma_n} \Sigma^\infty X^{\wedge n}$$

$$\underbrace{\pi_* \mathcal{F}_{\partial_*}(\Sigma^\infty X)}_{\text{shifted Lie alg}} \Rightarrow \underbrace{\pi_*(X)}_{\text{shifted Lie alg}}$$

Johnson/Arone-Mahowald

$$H_*(\partial_*; \mathbb{Z}) = \mathrm{sLie}_*$$

"unstable homotopy grps are stable homotopy grps + whitehead products"

Open Problem

Is this a SS of sLie-alg's?

Does $[-,-]$ on E_1 converge to Whitehead product?

products

$H_*(\partial_*; Z) = sLie_*$

shifted Lie operad

Rationally: (Walter)

$$\underbrace{\pi_* \mathcal{F}_{Lie}(\Sigma^\infty X)_{\mathbb{Q}}}_{\shortparallel \; sLie(H_*(X;\mathbb{Q}))} \Rightarrow \pi_*(X)_{\mathbb{Q}}$$

$L(X)$ ↑ DGL model for $X_{\mathbb{Q}}$

Goodwillie tower = filtration by bracket length.

p-adic Story? complete all spaces at p

$$\pi_* \mathcal{F}_{\partial_*}(\Sigma^\infty X) = ??$$

$H_* \mathcal{F}_{\partial_*}(\Sigma^\infty X)$

Z = CW Spectrum: AHSS $\pi_*(-)$

p-complete spectra:

$$C_*^{cell}(Z) \otimes \pi_*^s \Rightarrow \pi_*(Z)$$

Z minimal CW structure.

One cell for every generator of $H_*(Z)$ $H_*(X) = H_*(X; \mathbb{F}_p)$

$p=2$

L algebra over ∂_* $\bar{Q}^i(x) = 0 \quad i < |x|$

Lie-Power operations $\bar{Q}^i : H_*(L) \longrightarrow H_{*+i-1}(L)$

$H_*(X) = \mathbb{F}_2\{x_j\}_j$

(p>2 Kjaer)

Thm (Antolín-Camarena $p=2$)

runs over a basis of $sLie\, H_*(X)$

$$H_* \mathcal{F}_{\partial_*}(\Sigma^\infty X) = \mathbb{F}_2 \left\{ \underbrace{\bar{Q}^{i_1} \cdots \bar{Q}^{i_k}}_{i_j > 2 i_{j+1}} \overbrace{[x_{j_1}[x_{j_2}[\cdots [x_{j_{\ell-1}}, x_{j_\ell}]\cdots]}^{} \right\}$$

$\partial_{2^k \ell} \wedge \cdots$

$\partial_{2^k\ell} \wedge \cdots$

$$\pi_*^s\{Q^I[x_{j_1},[x_{j_2},\cdots[\quad]\cdots]]\} \underset{\text{AHSS}}{\Longrightarrow} \pi_* \mathcal{F}_{\partial_*}(\Sigma^\infty X) \underset{\text{GSS}}{\Longrightarrow} \pi_*(X)$$

Specialize to $X = S^n$ $\qquad [x,x] = 0$

$$\tilde{H}_*(S^n) = \mathbb{F}_2\{\iota_n\}$$

$$H_* \mathcal{F}_{\partial_*}(S^n) = \mathbb{F}_2\{Q^{i_1}\cdots Q^{i_k}\iota_n\} \quad 2^k$$

$i_k \geq n$

$i_j \geq 2i_{j+1}+1$

Arone–Mahowald

$D_i(S^n) \simeq *$

unless $i = 2^k$

EHP sequence $\quad \Omega^2 S^{2n+1} \xrightarrow{P} S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$

$$\begin{tikzcd}
\pi_{k+n+1} S^{2n+1} \arrow[r, "P"] & \pi_{k+n-1} S^n \\
\pi_k S^n \arrow[u, "E^{n+1}"] \arrow[ur, "{[-,\iota_n]}"'] &
\end{tikzcd}$$

$$\pi_{2n-1} S^n \longrightarrow \pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$$

HI

$$\boxed{\begin{tikzcd}
\Omega S^1 \arrow[r] & \Omega^2 S^2 \arrow[r] \arrow[d, "H"] & \Omega^3 S^3 \arrow[r] \arrow[d, "H"] & \Omega^4 S^4 \arrow[r] \arrow[d, "H"] & \cdots \Omega^n S^n \\
& \Omega^2 S^3 & \Omega^3 S^5 & \Omega^4 S^7 &
\end{tikzcd}}$$

EHPSS

EHPSS $\bigoplus_{m\geq 1} \pi_* S^{2m-1} \Rightarrow \pi_*^s$

$\bigoplus_{m=1}^{n} \pi_* S^{2m-1} \Rightarrow \pi_* S^n$

$Id \xrightarrow{E} \Omega\Sigma \xrightarrow{H} \Omega\Sigma Sq \qquad Sq(x) \Rightarrow x\wedge x$

$f(x) \quad f(x^2)$

$P_n(x) \longrightarrow P_n(\Omega\Sigma)(x) \longrightarrow P_{\lfloor n/2\rfloor}(\Omega\Sigma)(x)$

$P_n(\Omega F\Sigma)(x) = \Omega P_n(F)(\Sigma X)$

$$P_{2^k}(S^n) \longrightarrow \Omega P_{2^k}(S^{n+1}) \longrightarrow \Omega P_{2^{k-1}}(S^{2n+1})$$

$\bar{Q}^{i_1}\cdots\bar{Q}^{i_{k-1}}\iota_{2n+1} \longmapsto \begin{matrix}\bar{Q}^{i_1}\cdots\bar{Q}^{i_k}\iota_n \\ Q^{i_1}\cdots Q^{n}\iota_n\end{matrix} \xmapsto{i_k\geq n} \begin{cases}\bar{Q}^{i_1}\cdots\bar{Q}^{i_k}\iota_{n+1}, & i_k\geq n+1 \\ 0, & o/w\end{cases}$

$$\mathbb{D}_{2^{k-1}}(S^{2n+1}) \longrightarrow \mathbb{D}_{2^k}(S^n) \longrightarrow \Sigma^{-1}\mathbb{D}_{2^k}(S^{n+1}) \longrightarrow \Sigma^{-1}\mathbb{D}_{2^{k-1}}(S^{2n+1})$$

tells you that $\mathbb{D}_{2^k}(S^n)$ built

fm $\mathbb{D}_{2^{k-1}}(S^{2m+1})$

$m \geq n$

AHSS $\bigoplus_m \pi_* \mathbb{D}_{2^{k-1}}(S^{2m+1}) \Rightarrow \pi_* \mathbb{D}_{2^k}(S^n)$

$Q^{i_1}\cdots Q^{i_{k-1}} \qquad Q^{i_1}\cdots Q^{i_k}\iota_n$

$m = i_k$

$S^{-1} \leftarrow P_n^\infty \qquad \mathbb{D}_2(S^n)$

$n \to -\infty$

$n \to -\infty \qquad \mathbb{D}_{2^k}(S^n) \qquad S^{-k}$