An introduction to Surgery

Applied surgery series MPIM

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Fundamental questions in Surgery :

① Existence

Let X be a space. When is X homotopy equivalent to a closed manifold M ?

② Uniqueness

If $f: N \to M$ is a homotopy equivalence between closed manifolds, are they isomorphic?

"isomorphic" means
- Topological : homeomorphic
- Piecewise Linear : PL homeomorphic
- Smooth : Diffeomorphic

Answer to Q.2: No in general! But...

Theorem (Generalized Poincaré conjecture)

Let M be a closed manifold h. eq. to S^n. Then M is homeomorphic to S^n.

Conjecture (Borel).

Path connected & $\pi_n(M)$ is trivial for $n \geq 2$

Let M and N be closed aspherical manifolds.
If $M \underset{h.eq}{\simeq} N \implies M$ is homeomorphic to N.

Known to be true in many cases.

Conjecture (Hurewicz)

π_1 is trivial

Let M and N be closed simply-connected manifolds.

If $M \underset{h.eq}{\simeq} N \implies M$ is homeomorphic to N.

False!

Example There exists a manifold

$$E^7 \underset{h.eq}{\simeq} S^3 \times S^4$$

$\exists$ an oriented sphere bundle $S(\xi) : S^3 \hookrightarrow E \to S^4$ s.t. E is fiber homotopically trivial and $p_1(S(\xi)) \neq 0$.

homotopy equiv

$$\begin{tikzcd}
S^3 \times S^4 \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d] \\
S^4 \arrow[r, "id"] & S^4
\end{tikzcd}$$

But $S^3 \times S^4$ is parallelizable $\Rightarrow p_1^{\mathbb{Q}}(S^3 \times S^4) = 0$

Rational Pontrjagin classes are invariant under homeomorphism.

Novikov got the Fields medal in 1970 for this!

If h were a homeomorphism

$$h^*(p_i^{\mathbb{Q}}(E)) \neq p_i^{\mathbb{Q}}(\underbrace{S^3 \times S^4}_{=0}).$$

The fact that Pontryagin classes do not coincides allows us to deduce that E is not homeomorphic to $S^3 \times S^4$.

A good reference to look up details for this is Milnor-Stasheff around Lemma 20.10.

Novikov had a similar example for $S^4 \times S^5$.

A structure set

Definition: The structure set $\mathcal{S}_n(X)$ of X is the set of equivalence classes of pairs

$$(\underbrace{M}_{\boxed{\text{oriented } n\text{-dim closed manifold}}}\ ,\ \underbrace{f: M \to X}_{\boxed{\text{homotopy equivalence}}})$$

Two pairs $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are equivalent if there exists an orientation preserving homeomorphism $h : M_1 \to M_2$ s.t.

$$\begin{tikzcd}
M_1 \arrow[dr, "f_1"] \arrow[dd, "h"'] & \\
 & X \\
M_2 \arrow[ur, "f_2"'] &
\end{tikzcd}
\quad \circlearrowleft$$

Commutes up to homotopy.

Remarks

- To obtain a true classification one has to mod out by the self-homotopy equivalences of X.

- To have the structure set fit into the surgery exact sequence one has to be precise about :
  - Category
  - h- vs s-cobordism

Examples

(Poincaré conjecture) $M \xrightarrow[h.eq]{\simeq} S^n \Rightarrow M$ homeo S^n

$\hookrightarrow \mathcal{S}_3(S^3) = *, \quad \mathcal{S}_4(S^4) = *$

simply connected

(Hurewicz conj): $M \underset{h.eq}{\simeq} N \Rightarrow M$ homeo N

$\hookrightarrow \mathcal{S}_7(S^3 \times S^4) \neq *$

aspherical

(Borel conjecture) $M \underset{h.eq}{\simeq} N \Rightarrow M$ homeo N

$\hookrightarrow \mathcal{S}_n(N) = *$, N aspherical

Recall the existence question:

$$\boxed{\text{When is } M \xrightarrow[h.eq]{\simeq} X \quad ?}$$

The importance of bundle data.

X is a Poincaré complex

CW, Poincaré duality

$-\cap [X] : H^{n-*}(X) \to H_*(X)$

Theorem: Every Poincaré complex X admits a "Spivak normal fibration".

... but not always a normal bundle.

The primary obstruction to surgery is if the Spivak normal fibration does not admit a vector bundle reduction

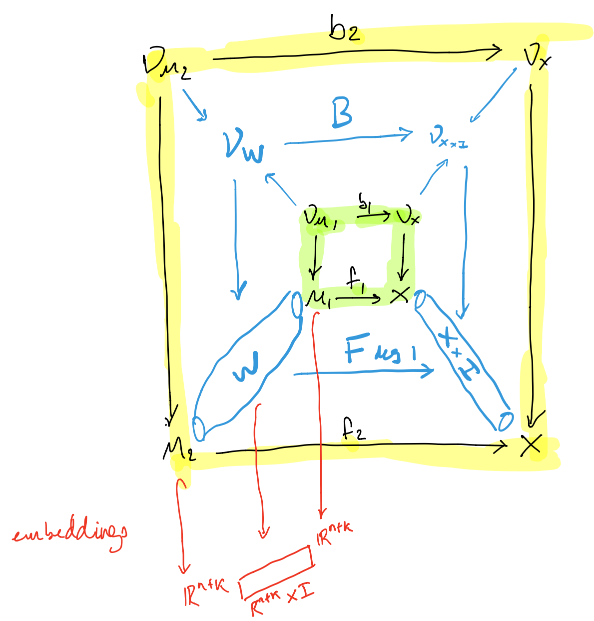
Def A degree 1 normal map with target X consists of:

$$\begin{tikzcd} \nu_M \arrow[r, "b"] \arrow[d] & \nu_X \arrow[d] \\ M \arrow[r, "f", "\text{degree } 1"'] & X \end{tikzcd}$$

→ vector bundle reduction of Spivak normal bundle

$$H_n(M) \longrightarrow H_n(X)$$
$$[M] \longmapsto f_*[M] = [X]$$

Two degree 1 normal maps are cobordant if $\exists$ a "cobordism" on all the structure:



The green and the yellow are two deg 1 normal maps.

The blue in this diagram gives an idea of what a cobordism is between them.

Denote by $N(X)$ the set of cobordism classes of degree 1 normal maps.

The surgery step

Recall the existence question:

$$\boxed{\text{When is } M \xrightarrow[h.eq]{\simeq} X \ ?}$$

Want: A procedure so that we can modify

$$\begin{tikzcd} M \arrow[rr, squiggly] \arrow[dr, "f"'] & & \boxed{M'} \arrow[dl, "f'"] \\ & X & \end{tikzcd} \quad \text{homotopy equivalence}$$

without changing X.

What does a map f need to satisfy in order for it to be a h.eq ?

Whitehead Theorem: f is a h.eq iff it is k-connected for all $k \geq 0$.

Recall: k-connected means that

$$\pi_j(M, m) \longrightarrow \pi_j(X, f(m)) \text{ is } \begin{cases} \text{bijective for } j < k \\ \qquad \text{and} \\ \text{surj for } j = k \end{cases}$$

Recall there is a long exact homotopy sequence

$$\dashrightarrow \pi_{k+1}(M) \to \pi_{k+1}(X) \to \pi_{k+1}(f) \to \pi_k(M) \to \pi_k(X)$$

$$\pi_{k+1}(f) = \pi_{k+1}(\mathrm{Cyl}(f), X)$$

want this to be trivial

consists of homotopy classes of commutative squares

$$\begin{tikzcd} S^k \arrow[r, "q"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "f"] \\ D^{k+1} \arrow[r, "Q"] & X \end{tikzcd}$$

suppose f is k-connected (ie $\pi_j(f) = 0$ for $j \leq k$)

we need to achieve $\pi_{k+1}(f) = 0$ without changing $\pi_j(f)$ for any $j \leq k$

Consider the pushout :

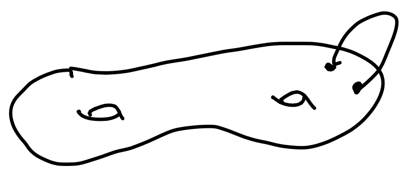
$$\begin{tikzcd}
S^k \arrow[r, "g"] \arrow[d, hook, "i"'] & M \arrow[d] \arrow[ddrr, bend left, "f"] & & \\
D^{k+1} \arrow[r] \arrow[drrr, bend right] & M \cup_{S^k} D^{k+1} \arrow[drr, "f'"] & & \\
& & & X
\end{tikzcd}$$

$f' : M \cup D^{k+1} \longrightarrow X$ is obtained from $f : M \to X$ by attaching a cell.

This has the desired effect on homotopy.

Problem

Attaching a cell destroys the manifold structure.



idea : instead of starting with

we extend to :

In general, we can only expect this to be an immersion.

$$\begin{tikzcd} S^k \times D^{n-k} \arrow[r, "q"] \arrow[d, hook, "i"'] & M^n \arrow[d, "f"] \\ D^{k+1} \times D^{n-k} \arrow[r, "Q"] & X \end{tikzcd}$$

when $n=2k$ there will be a surgery obstruction if we are not able to get rid of the self-intersections and convert the immersion into an embedding.

During the talk, I made a comment here about the importance of the bundle data to ensure that the thickening exists.

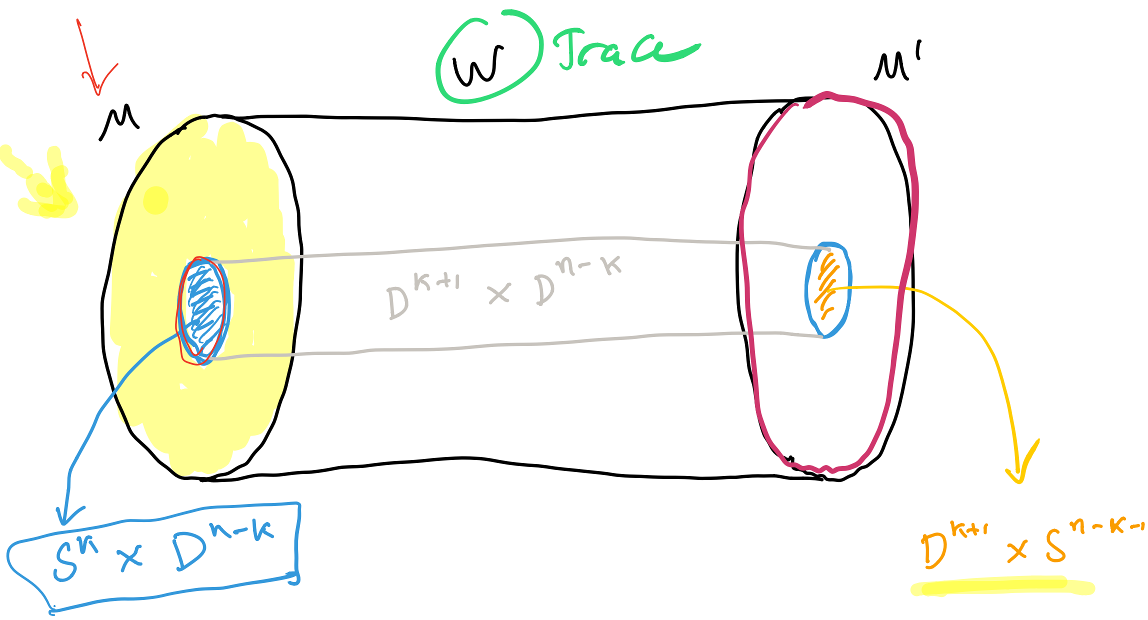
Input : Manifold M^n and an embedding

$$S^k \times D^{n-k} \subset M$$

Output : New n-dim manifold M'

$$M' = \text{closure}\left(M \setminus S^k \times D^{n-k}\right) \cup D^{k+1} \times S^{n-k-1}$$

$$\partial\left(S^k \times D^{n-k}\right) = S^k \times S^{n-k-1} = \partial\left(D^{k+1} \times S^{n-k-1}\right)$$

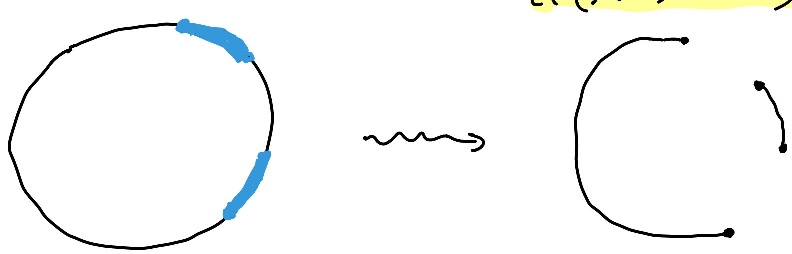


Closed manifolds are cobordant iff they can be obtained from one another from a sequence of surgeries.

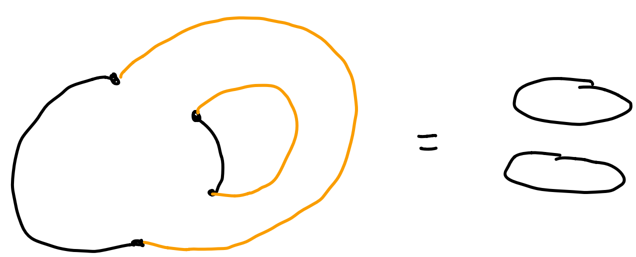
Example

$M = S^1$ 0-surgery : $S^0 \times D^1 \subset S^1$

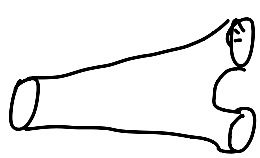
M $\qquad$ $cl.(M \setminus S^0 \times D^1)$



Glue $D^1 \times S^0$



$$M' = cl.(M \setminus S^0 \times D^1) \cup D^1 \times S^0$$



Surgery exact sequence

If $\mathcal{S}(X) \neq \phi$

$$\cdots L_{n+1}(\mathbb{Z}\pi, X) \longrightarrow \underset{\{f: M \to X\}/}{\underline{\mathcal{S}^{CAT}(X)}} \longrightarrow \underset{(f,b)}{\mathcal{N}^{CAT}(X)} \longrightarrow L_n(\mathbb{Z}\pi, X)$$

surgery obstruction group

The Surgery Exact Sequence is more powerful in the study of the uniqueness question. (The X in the statement would be a manifold in that case).