Applied surgery 3

Topological rigidity and low dim top

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Aspherical Spaces

X aspherical if $\tilde{X} \simeq *$ i.e. $X = K(\pi, 1)$

e.g.  

- homotopy rigidity: X, Y asph.

$$X \simeq Y \iff \pi_1 X \cong \pi_1 Y$$

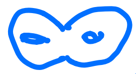
$$\dim X < \infty \Rightarrow \pi_1 X \text{ torsion free}$$

Examples: closed Aspherical Manifolds

X^n closed

$n = 0$ $\quad \cdot$

$n = 1$ 

$n = 2$ , , $T^2 \# \cdots \# T^2$, $K \# T^2 \# \cdots \# T^2$

$n = 3$ $\quad X$ asph $\iff$ X irr & $\pi_1 X$ inf.

$n = 4$ $\quad T^4, \ldots$

Examples: cpt asph mflds w boundary

$n = 1$ 

$n = 2$ , , closed 2-mfld − open disks

$n = 3$ D^3, knot exterior

$n = 4$, D^4, $S^1 \times D^3$, $T^2 \times D^2$, $T^3 \times D^1$, T^4

Relative Structure Set

(X, M)

space, closed manifold

$$\mathcal{S}(X \text{ rel } M) = \mathcal{S}_M(X) = \mathcal{S}_\partial(X)$$

$$[(N, \partial N) \xrightarrow{h} (X, M)] \in \mathcal{S}$$

cpt mfld

$N \xrightarrow{\simeq} X$, $\partial N \xrightarrow{\cong} M$

$h \sim h'$ if

$$\begin{tikzcd} N \arrow[d, "\cong"'] \arrow[dr, "h"] & \\ N' \arrow[r, "h'"'] & X \end{tikzcd}$$

commutes up to htpy rel ∂N

Borel Conj – Top Rigidity

$$\mathcal{S}(X \text{ rel } M) = pt$$

	uniqueness	existence
closed	$\mathcal{S}(\text{closed asp mfld}) = *$	$\mathcal{S}(\text{asp Poin complex}) = *$
cpt w ∂	$\mathcal{S}_\partial(\text{cpt asp mfld}) = *$ $\cancel{\checkmark}$	(X, M) Poin pair, M closed mfld, X asp cplx

$$\mathcal{S}(X) = \{Id_X\} \iff \begin{cases} Y \text{ closed asp}, \ \pi_1 Y \cong \pi_1 X \Rightarrow X \cong Y \\ h : X \xrightarrow{\simeq} X \Rightarrow h \simeq \text{homeo} \end{cases}$$

An Example: 4d, simply connected

BC for $(D^4, S^3) \Rightarrow$ 4d Poincare conj

$X^4 \simeq * \Rightarrow \partial X$ $\mathbb{Z}HS$

BC for $\pi_1 = 1 \Leftrightarrow$

Thm (Freedman): Any $\mathbb{Z}HS$ bounds unique contractible cpt 4-mfld

Thm (D-H) Any $\mathbb{Z}[\mathbb{Z}^k]$-homology $T^k \times S^{n-k-1}$ bounds unique asph mfld with $\pi_1 = \mathbb{Z}^k$.

$$H_1 M^{n-1} = \mathbb{Z}^k$$

$$\begin{array}{c} \overline{M} \\ \downarrow \mathbb{Z}^k \\ M \end{array}$$

$$H_* \overline{M} = H_*(S^{n-k-1})$$

$$\Longrightarrow \quad M = \partial X \qquad X \simeq K(\mathbb{Z}^k, 1)$$

Farrell-Jones Conj (FJC)

FJC for π torsion free

$$A : H_*(B\pi; \underline{L}) \xrightarrow{\cong} L_*(\mathbb{Z}\pi)$$

$$\begin{array}{ccccccc} L_{n+1}(\mathbb{Z}\pi) \to & \mathcal{S}_\partial(X) & \longrightarrow & \mathcal{N}_\partial(X) & \xrightarrow{\sigma} & L_n(\mathbb{Z}\pi) \\ & & & \| & & \uparrow A \\ & & & H_n(X; \underline{L}) & \longrightarrow & H_n(B\pi; \underline{L}) \end{array}$$

$\therefore$ For X not asph. (eg $S^3 \times S^4$) expect $\mathcal{S} \neq pt$

For X asph, expect $\mathcal{S} = *$

$$\pi_i \, \underline{L} = L_i(\mathbb{Z}) = \begin{cases} \mathbb{Z} & i \equiv 0 \, (4) \\ \mathbb{Z}/2 & i \equiv 2 \, (4) \end{cases}$$

$$H_n(X; \underline{L}) = \pi_n(X_+ \wedge \underline{L})$$

FJC implies BC

(X, M) — X asph, M closed

$\dim X \geq 5$

closed unique.

FJC for $\pi_1 X$? exist.

	unique	exist
closed	FJC $\Rightarrow$ BC	FJC $\not\Rightarrow$ BC
cpt w $\partial \neq 0$	FJC $\Rightarrow$ BC	FJC $\Rightarrow$ BC DH

$$\begin{tikzcd}
\mathcal{N}_\partial X \arrow[r, "\sigma"] \arrow[d, "\cong"] & L_n(\mathbb{Z}\pi, w) \\
H_n(X; \mathbb{L}\langle 1\rangle) \arrow[d] & \\
H_n(X; \mathbb{L}) \arrow[r] & H_n(B\pi; \mathbb{L}) \arrow[uu, "\cong"', "A"]
\end{tikzcd}$$

Poincare duality pairs (PDn pairs)

PD_4-pair

Thm (DH) (X, M^{n-1})

asph CW, dim $= n$ — closed

If $H^i(X, M; \mathbb{Z}\pi) = \begin{cases} 0 & i \neq n \\ \mathbb{Z} & i = n \end{cases}$

then (X, M) is Poincare pair.

<u>Cor:</u> FJC for π $\Rightarrow$ $\exists$ $K(\pi, 1)$-mfld with $\partial = M$

Surgery in 4d

- topological cat. only
- may not work for $\pi_1 = \pi_1^\infty = F(2)$
- works for EA groups
- ~~FJC holds for all EA groups~~

~~Cor: Borel-Uniqueness holds for $\pi_1 X = EA$~~

Thm (DH) BC-Existence holds for PD_4 pairs with π EA

Elementary Amenable (EA) groups

Def: EA groups are the smallest class of groups

- containing finite & abelian groups
- closed under extensions, subquotient, direct limits

e.g. finite, abelian, polycyclic $\pi_1(KB)$

Baumslag-Solitar $BS(1,m) = \mathbb{Z}[\frac{1}{m}] \rtimes_m \mathbb{Z}$

EA Fund. groups of aspherical 4-mflds

Thm (DH): EA π is π_1(cpt asp 4-mfd) iff

- $\pi = 1$ $\quad M^3$ Poincaré hom sphere $\Longrightarrow$ $M^3 = \partial X^4$ ← top mfld
- $\pi = \mathbb{Z}$
- $\pi = \mathbb{Z}[1/m] \rtimes_m \mathbb{Z} = BS(1,m)$ $\quad K \times D^2$
- π is polycyclic PD_3 $\quad T^3 \times I$
- π is polycyclic PD_4, $\partial X = \emptyset$

<u>Cor:</u> π satisfies FJC, hence BC.

High dim'l surgery in 3d

SES works with $\mathcal{S}^H$!

$$L_4(\mathbb{Z}\pi_1 X) \longrightarrow \mathcal{S}^H(X^3) \longrightarrow \mathcal{N}(X) \longrightarrow L_3(\mathbb{Z}\pi_1 X)$$

$$\mathcal{S}^H(X^3) \ni \; N \to X \quad \mathbb{Z}\pi_1 X\text{-hom eq.}$$

e.g. $\underset{\infty}{\mathcal{S}^H(\text{lens space})} \longrightarrow \underset{\text{finite}}{\mathcal{S}(\text{lens space})}$

Existence: $\forall$ finite G, $\exists$ free G-actn on $\mathbb{Q}HS$ M^{4k+3}. $\quad (L_3(\mathbb{Q}G) = 0)$

$$\begin{tikzcd}
\bar{N} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
N \arrow[r] & Y
\end{tikzcd}$$

homol equiv