Towards a construction of limits in $(\infty,2)$-categories

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2-dimensional categories

$$\text{Category: } \left\{\begin{array}{lll} \text{objects} & \longrightarrow & \text{enriched in Set} \\ \text{morphisms} & \longrightarrow & \text{internal to Set} \end{array}\right.$$

2-category

objects $\bullet$

$$\left.\begin{array}{ll} \text{morphisms} & \bullet \longrightarrow \bullet \\ \text{2-morphisms} & \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd} \end{array}\right\}\text{category}$$

$\longrightarrow$ enriched in Cat

Double category

objects $\bullet$

horizontal morphisms $\bullet \longrightarrow \bullet$

vertical morphisms $\begin{tikzcd} \bullet \arrow[d, "\shortmid" marking] \\ \bullet \end{tikzcd}$

squares

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\shortmid" marking] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "\shortmid" marking] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

$\longrightarrow$ internal to Cat

2-category $\mathcal{A}$ $\rightsquigarrow$ horizontal double category $\mathbb{H}\mathcal{A}$
with only trivial vertical morphisms

$\rightsquigarrow$ Often too strict to accomodate many examples

Higher categories

(∞, n)-category

Great tools to prove theorems in several areas of mathematics

objects
morphisms
⋮
n-morphisms
(n+1)-morphisms
⋮

$(\infty, n-1)$-category

(n+1)-morphisms, ⋮ : invertible

Compositions are associative only up to higher invertible morphisms

→ enriched in $(\infty, n-1)$-categories.

Double $(\infty,1)$-categories: internal to $(\infty,1)$-categories

$(\infty,2)$-categories: double $(\infty,1)$-categories with only trivial vertical morphisms

⇝ To define the ∞-structures, we need models

Models of (∞,n)-categories

Higher categories can be modelled via model categories in which they are the fibrant objects.

$\underline{n=0}$: ∞-groupoids: – Topological spaces
– Kan complexes [Quillen]

$\underline{n=1}$: $(\infty,1)$-categories: – Quasi-categories [Joyal]
– Kan-enriched categories [Bergner]
– Marked simplicial sets [Lurie]
– Complete Segal spaces [Rezk]
⋮

$\underline{n=2}$: $(\infty,2)$-categories: – Quasi-2-categories [Ara]
– qCat-enriched categories [Lurie]
– Scaled simplicial sets [Lurie]
– 2-fold complete Segal spaces [Barwick]
– Θ_2-spaces [Rezk]
⋮

Categorical constructions in the ∞-world

$\left.\begin{array}{l}(\infty,1)\text{-categories}\\(\infty,2)\text{-categories}\end{array}\right\}$ should be thought of as homotopical versions of $\left\{\begin{array}{l}\text{categories}\\\text{2-categories}\end{array}\right.$

$\rightsquigarrow$ Most constructions and theorems of (2-)category theory should hold in the ∞-setting (only up to homotopy)

Already developed in the $(\infty,1)$-categorical case: for example, limits in an $(\infty,1)$-category have been constructed and studied in different models of $(\infty,1)$-categories [Joyal, Lurie, Riehl-Verity, Heuts-Moerdijk, Rasekh, Gepner-Haugseng-Nikolaus, Rovelli, ...]

Develop a useable theory of $(\infty,2)$-categories

Work in progress with Nima Rasekh and Martina Rovelli: construct a notion of limits in an $(\infty,2)$-category in a model-independent way.

[Gagna-Harpaz-Lanari]: Limits in ∞-bicategories

Limits

Limit of a functor F btw categories $\leftrightsquigarrow$ Terminal object in the category $\Delta\downarrow F$ of cones over F

Limit of a functor f btw $(\infty,1)$-categories $\overset{\text{def}}{\leftrightsquigarrow}$ Terminal object in the $(\infty,1)$-category $\Delta\downarrow f$ of cones over f

2-limit of a 2-functor F btw 2-categories $\not\leftrightsquigarrow$ 2-terminal object in the 2-category $\Delta\downarrow F$ of cones over F [Clingman-M.]

$\leftrightsquigarrow$ Double terminal object in the double category $\Delta\downarrow \mathbb{H}F$ of cones over $\mathbb{H}F$ [Grandis-Paré, Clingman-M.]

Examples of 2-limits:

- Grothendieck construction (lax colimit)
- Category of algebras for a 2-monad (lax limit)

Relations btw strict and ∞-settings

Need model structures in strict setting to compare with ∞-setting

[Lack] ∃ model structure on 2Cat (w.e. = biequivalences)

[Fiore-Paoli-Pronk]: ∃ model structures on DblCat

⇝ not compatible with $\mathbb{H}: 2\mathrm{Cat} \to \mathrm{DblCat}$

[M.-Sarazola-Verdugo]: ∃ model structure on DblCat such that $\mathbb{H}: 2\mathrm{Cat} \to \mathrm{DblCat}$ is a homotopically full embedding

[M.] ∃ homotopically compatible square

$$\begin{tikzcd}
{[\text{Lack}]} \arrow[r] & 2\mathrm{Cat} \arrow[r, "\mathbb{N}\mathbb{H}"] \arrow[d, "\mathbb{H}"'] & \mathrm{Cat}_{(\infty,2)} \arrow[d] & \text{2-fold complete Segal spaces [Barwick]} \arrow[l] \\
{[\text{MSV}]} \arrow[r] & \mathrm{DblCat} \arrow[r, "\mathbb{N}"'] & \mathrm{DblCat}_{(\infty,1)} & \text{Segal objects in complete Segal spaces [Haugseng]} \arrow[l]
\end{tikzcd}$$

Other nerves $2\mathrm{Cat} \to \mathrm{Cat}_{(\infty,2)}$: [Ozornova-Rovelli] (2-precomplicial sets); [Campbell] (quasi-2-categories); [Gagna-Harpaz-Lanari] (∞-bicategories)

⇝ [M.-Ozornova-Rovelli]: comparison btw nerves

Defining limits in an $(\infty,2)$-category

Other characterization of 2-limits [Clingman-M.]

2-limit of a 2-functor F btw 2-categories $\leftrightsquigarrow$ 2-terminal object in the "shifted" 2-category of cones $\mathrm{Ar}_*\Delta\downarrow F$

"Shifted" version is constructed using the functor

$$\mathrm{Ar}_*: 2\mathrm{Cat} \xrightarrow{\mathbb{H}} \mathrm{DblCat} \xrightarrow{[\mathbb{V}[1],-]} \mathrm{DblCat} \xrightarrow{\mathbf{H}} 2\mathrm{Cat}$$

$\mathbb{H} \dashv \mathbf{H}$

(Ar_* is the base change functor along $\mathrm{Ar}: \mathrm{Cat} \to \mathrm{Cat}$, $\mathcal{C} \mapsto \mathcal{C}^{[1]}$)

Adaptable to the ∞-setting [M.-Rasekh-Rovelli]

$$\mathrm{Ar}_*^\infty: \mathrm{Cat}_{(\infty,2)} \xrightarrow{\mathbb{H}^\infty} \mathrm{DblCat}_{(\infty,1)} \xrightarrow{[\mathbb{NV}[1],-]} \mathrm{DblCat}_{(\infty,1)} \xrightarrow{\mathbf{H}^\infty} \mathrm{Cat}_{(\infty,2)}$$

[Haugseng] $\quad \Theta_2$-spaces [Rezk]

Limit in an $(\infty,2)$-category: terminal object in the "shifted" $(\infty,2)$-category of cones defined using Ar_*^∞

- Define limits this way in several models and compare them.
- Add weight $\rightsquigarrow$ lax limits.