From Archimedean L-factors to Topological Field Theories *

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Introduction

Archimedean local L-factors were introduced to simplify functional equations of global L-functions. From the point view of arithmetic geometry these factors complete the Euler product representation of global L-factors by taking into account Archimedean places of the compactified spectrum of the global field. A construction of non-Archimedean local L-factors is rather transparent and uses characteristic polynomial of the image of the Frobenius homomorphism in finite-dimensional representations of the local Weil-Deligne group closely related to the local Galois group. On the other hand, Archimedean L-factors are expressed through products of Γ-functions and thus are analytic objects avoiding simple algebraic interpretation. Moreover, Archimedean Weil-Deligne groups are rather mysterious objects in comparison with their non-Archimedean counterparts. In a series of papers [GLO1], [GLO2], [GLO3], [GLO4] we approach the problem of the proper interpretation of Archimedean L-factors using various methods developed to study quantum integrable systems and low-dimensional topological field theories. As a result we produce several interesting explicit representations for Archimedean L-factors and related special functions revealing some hidden structures that might be relevant to the Archimedean (also known as ∞-adic) algebraic geometry. Some of our considerations are close to the approach advocated by Deninger [D1], [D2]. Also equivariant symplectic volumes of the space of maps of a disk into symplectic manifolds were previously discussed in [Gi1], [Gi2] in connection with the Gromov-Witten theory.

1 Archimedean Hecke algebra

Let K be a maximal compact subgroup of $G = GL(\ell+1,\mathbb{R})$. Define spherical Hecke algebra $\mathcal{H}_{\mathbb{R}} = \mathcal{H}(GL(\ell+1,\mathbb{R}),K)$ as an algebra of K-biinvariant functions on G, $\phi(g) = \phi(k_1 g k_2)$, $k_1, k_2 \in K$ with the multiplication given by

$$\phi * f(g) = \int_G \phi(g\tilde{g}^{-1})\, f(\tilde{g}) d\tilde{g}. \tag{1.1}$$

To ensure the convergence of the integrals one usually imposes the condition of compact support on K-biinvariant functions. We will consider a more general class of exponentially decaying functions.

By the multiplicity one theorem for principle series representations of $GL(\ell+1,\mathbb{R})$ there is a unique smooth spherical vector $\langle k|$ in a principal series irreducible representation $\mathcal{V}_{\underline{\lambda}} = \mathrm{Ind}_{B_-}^G\, \chi_{\underline{\lambda}}$ where $\chi_{\underline{\lambda}}$ is a character of a Borel subgroup B_-. The action of a K-biinvariant function ϕ on the spherical vector $\langle k|$ in $\mathcal{V}_{\underline{\lambda}}$ is reduced to multiplication by a character Λ_ϕ of the Hecke algebra:

$$\phi * \langle k| \equiv \int_G dg\phi(g^{-1})\, \langle k|\pi_{\underline{\lambda}}(g) = \Lambda_\phi(\underline{\lambda})\langle k|, \qquad \phi \in \mathcal{H}_{\mathbb{R}}. \tag{1.2}$$

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Define $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}$ as a matrix element in a principle series irreducible representation $\mathcal{V}_{\underline{\lambda}}$ satisfying the covariance property

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(kan) = \chi_N(n)\,\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(a), \tag{1.3}$$

where $kan \in KAN_- \to G$ is the Iwasawa decomposition. We parametrize the representations $\mathcal{V}_{\underline{\lambda}}$ of $GL(\ell+1,\mathbb{R})$ by vectors $\underline{\lambda} = (\lambda_1, \cdots, \lambda_{\ell+1})$ in $\mathbb{C}^{\ell+1}$. Whitaker functions play an important role in the theory of quantum integrable systems providing explicit solutions of quantum Toda chains. Let us define a related function

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}) = e^{-\langle\rho,\underline{x}\rangle}\;\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}), \tag{1.4}$$

where $\underline{x} = (x_1, \ldots, x_{\ell+1}) \in \mathbb{R}^{\ell+1}$, $\rho \in \mathbb{R}^{\ell+1}$, with $\rho_j = \frac{\ell}{2} + 1 - j,\;\; j = 1, \ldots, \ell+1$ and we use the standard orthogonal pairing $\langle\,,\,\rangle$ on $\mathbb{R}^{\ell+1}$. The functions (1.4) are common eigenfunctions of a ring of commuting differential operators generated by coefficients of a polynomial

$$t^{\mathfrak{gl}_{\ell+1}}(\lambda) = \sum_{j=1}^{\ell+1} (-\imath)^j \lambda^{\ell+1-j} \mathcal{H}^{\mathfrak{gl}_{\ell+1}}_j(x, \partial_x), \tag{1.5}$$

where the first two operators are given by

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_1 = -\imath \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_i}, \qquad \mathcal{H}^{\mathfrak{gl}_{\ell+1}}_2 = -\frac{1}{2}(\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_1)^2 - \frac{1}{2}\sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial {x_i}^2} + \sum_{i=1}^{\ell} e^{x_i - x_{i+1}}. \tag{1.6}$$

The last differential operator is a quantum Hamiltonian operator of $\mathfrak{gl}_{\ell+1}$-Toda chain. Commuting differential operators (1.5) provide an action of the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_{\ell+1})$ on the matrix elements satisfying (1.3). We have

$$t^{\mathfrak{gl}_{\ell+1}}(\lambda)\;\; \Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}) = \prod_{j=1}^{\ell+1}(\lambda - \lambda_j)\;\; \Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}). \tag{1.7}$$

The following version of the Givental integral representation [Gi3] for $\mathfrak{gl}_{\ell+1}$-Whittaker function was proposed in [GKLO].

Theorem 1.1 *The following integral recursive representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions holds*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(\underline{x}_{\ell+1}) = \int_{\mathbb{R}^\ell} \prod_{i=1}^{\ell} dx_{\ell,i}\; Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1}, \underline{x}_\ell|\lambda_{\ell+1}) \Psi^{\mathfrak{gl}_\ell}_{\lambda_1,\ldots,\lambda_\ell}(\underline{x}_\ell), \tag{1.8}$$

$$Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1}, \underline{x}_\ell|\lambda_{\ell+1}) = \exp\left\{\lambda_{\ell+1}\Big(\sum_{i=1}^{\ell+1} x_{\ell+1,i} - \sum_{i=1}^{\ell} x_{\ell,i}\Big) - \sum_{i=1}^{\ell}\Big(e^{x_{\ell+1,i}-x_{\ell,i}} + e^{x_{\ell,i}-x_{\ell+1,i+1}}\Big)\right\},$$

where $\underline{x}_k = (x_{k,1}, \ldots, x_{k,k})$ and we assume that $Q^{\mathfrak{gl}_1}_{\mathfrak{gl}_0}(x_{11}|\lambda_1) = e^{\lambda_1 x_{1,1}}$.

Note that due to (1.2) any left K-invariant matrix element is an eigenfunction with respect to the action of any $\phi \in \mathcal{H}_{\mathbb{R}}$. Thus we have for the Whittaker function

$$\phi * \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g) = \Lambda_\phi(\underline{\lambda})\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \qquad \phi \in \mathcal{H}_{\mathbb{R}}, \tag{1.9}$$

Theorem 1.2 *Let $\phi_{\mathcal{Q}_B(\lambda)}(g)$ be a K-biinvariant function on $G=GL(\ell+1,\mathbb{R})$ given by*

$$\phi_{\mathcal{Q}_B(\lambda)}(g)=2^{\ell+1}|\det g|^{\lambda+\frac{\ell}{2}}e^{-\pi\mathrm{Tr}g^tg}. \tag{1.10}$$

Then, the action of $\phi_{\mathcal{Q}_0(\lambda)}$ on the Whittaker function $\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g)$ (defined by (1.3)) descends to the action of an integral operator $\mathcal{Q}_B^{\mathfrak{gl}_{\ell+1}}(\lambda)$ with the kernel

$$\mathcal{Q}_B^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{y}|\lambda)=2^{\ell+1}\exp\Big\{\sum_{j=1}^{\ell+1}(\lambda+\rho_j)(x_j-y_j)-\pi\sum_{k=1}^{\ell}\Big(e^{2(x_k-y_k)}+e^{2(y_k-x_{k+1})}\Big)-\pi e^{2(x_{\ell+1}-y_{\ell+1})}\Big\},$$

where $\underline{x}=(x_1,\dots,x_{\ell+1})$ and $\underline{y}=(y_1,\dots,y_{\ell+1})$. The corresponding eigenvalue

$$\big(\phi_{\mathcal{Q}_B(\lambda)}*\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}\big)(g)=L_{\mathbb{R}}(\lambda|\underline{\lambda})\;\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \tag{1.11}$$

is given by

$$L_{\mathbb{R}}(\lambda|\underline{\lambda})=\prod_{j=1}^{\ell+1}\pi^{-\frac{\lambda-\lambda_j}{2}}\Gamma\Big(\frac{\lambda-\lambda_j}{2}\Big). \tag{1.12}$$

The integral operator $\mathcal{Q}_B^{\mathfrak{gl}_{\ell+1}}(\lambda)$ is an example of the Baxter operator which provides a key tool to solve quantum integrable systems. Its construction for quantum $\mathfrak{gl}_{\ell+1}$-Toda chains and its interpretation as an element of a spherical Hecke algebra $\mathcal{H}_{\mathbb{R}}$ was given in [GLO1].

The eigenvalues (1.12) can be considered as elementary building blocks from which general Whittaker functions can be constructed via Mellin-Barnes representations. Consider a simple example of the degenerate Whittaker function for which an analog of the Givental representation is given by

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(x)\;=\;\int\limits_{\mathbb{R}^\ell}\prod_{k=1}^{\ell}dx_{k,1}\quad e^{\mathcal{F}(x_{1,1},\dots,x_{\ell,1},x_{\ell+1,1})}, \tag{1.13}$$

where $x:=x_{\ell+1,1}$ and

$$\mathcal{F}(t)\;=\;\lambda_1x_{11}+\sum_{k=1}^{\ell}\lambda_{k+1}(x_{k+1,1}-x_{k,1})\;-\;e^{x_{11}}-\sum_{k=1}^{\ell}e^{x_{k+1,1}-x_{k,1}}.$$

The degenerate Whittaker function satisfies the following differential equation

$$\Big\{\prod_{k=1}^{\ell+1}\Big(-\frac{\partial}{\partial x}+\lambda_k\Big)\;-\;e^x\Big\}\;\Psi_{\underline{\lambda}}(x)\;=\;0. \tag{1.14}$$

Besides the Givental representation there exists a representation of the Mellin-Barnes type

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(x)\;=\;\int\limits_{\sigma-\imath\infty}^{\sigma+\imath\infty}d\lambda\;e^{\lambda x}\;\prod_{k=1}^{\ell+1}\Gamma\left(\lambda_k-\lambda\right), \tag{1.15}$$

where σ is such that $\sigma<\min\{\,\mathrm{Re}\,\lambda_j, j=1,\dots,\ell+1\}$. Thus, basically, the degenerate Whittaker function is given by an action of integral projection operator on a product of eigenvalues (1.12).

There is a p-adic analog $\mathcal{H}_p=\mathcal{H}(GL(\ell{+}1,\mathbb{Q}_p),GL(\ell{+}1,\mathbb{Z}_p))$ of the Hecke algebra $\mathcal{H}_{\mathbb{R}}$. One can define a $\mathcal{H}_p$-valued function of an axillary variable such that its action by convolution on the p-adic analog [CS] of the Whittaker function is given by the multiplication on a local non-Archimedean

L-factor $L_p(s)$. In [GLO1] we argue that (1.10) should be considered as an Archimedean analog of the $\mathcal{H}_p$-valued function in non-Archimedean case. In particular the corresponding eigenvalues (1.12) are given by real Archimedean L-factors

$$L_{\mathbb{R}}(s|V,\Lambda) = \det_V \, \pi^{-\frac{s-\Lambda}{2}} \, \Gamma\left(\frac{s-\Lambda}{2}\right), \tag{1.16}$$

where $V = \mathbb{C}^{\ell+1}$, $s = \lambda$ and Λ is diagonal matrix with the diagonal entries $\Lambda_j = \lambda_j$. In the next Section we provide a functional integral representation of the Archimedean L-factors (1.16). Taking into account that general Whittaker functions can be constructed form L-factors this leads to a functional integral representation of general Whittaker functions.

2 L-factors via equivariant topological linear sigma model

In this Section we demonstrate how local Archimedean L-factors (1.16) can be described in the framework of the two-dimensional topological field theory. Precisely, we consider equivariant version of type A topological linear sigma model on a disk $D = \{z| \, |z| \leq 1\}$ with non-compact target space $X = \mathbb{C}^{\ell+1}$. The vector space $\mathbb{C}^{\ell+1}$ is supplied with a Kähler form and a Kähler metric given in local complex coordinates $(\varphi^j, \varphi^{\bar{j}})$ by

$$\omega = \frac{\imath}{2} \sum_{j=1}^{\ell+1} d\varphi^j \wedge \bar{\varphi}^j, \qquad g = \frac{1}{2} \sum_{j=1}^{\ell+1} (d\varphi^j \otimes d\bar{\varphi}^j + d\bar{\varphi}^j \otimes d\varphi^j). \tag{2.1}$$

We also supply the disk D with the flat metric $d^2s = dzd\bar{z} = dr^2 + r^2 d\sigma^2$, $z = r\, e^{\imath\sigma}$. Let K and $\bar{K}$ be canonical and anti-canonical bundles on D. Let $\mathrm{Map}(D, \mathbb{C}^{\ell+1})$ be the space of maps $\Phi : D \to X$ of the disk D to $\mathbb{C}^{\ell+1}$. Let $T_{\mathbb{C}}X = T^{1,0}\mathbb{C}^{\ell+1} \oplus T^{0,1}\mathbb{C}^{\ell+1}$ be a decomposition of the complexified tangent bundle of $\mathbb{C}^{\ell+1}$. Now let us specify the field content of the topological sigma model for $X = \mathbb{C}^{\ell+1}$. We define commuting fields F and $\bar{F}$ as sections of $K \otimes \Phi^*(T^{0,1}X)$ and of $\bar{K} \otimes \Phi^*(T^{1,0}X)$ correspondingly. The anticommuting fields χ, $\bar{\chi}$ are sections of the bundles $\Phi^*(\Pi T^{1,0}X)$, $\Phi^*(\Pi T^{0,1}X)$ and anticommuting fields ψ, $\bar{\psi}$ are sections of the bundles $K \otimes \Phi^*(\Pi T^{0,1}X)$, $\bar{K} \otimes \Phi^*(\Pi T^{1,0}X)$. Here $\Pi\mathcal{E}$ denotes the vector bundle $\mathcal{E}$ with the reverse parity of the fibres. Denote by $\langle , \rangle$ a natural Hermitian pairing on the spaces of sections of various bundles involved. We have the standard action of $U_{\ell+1}$ on $V = \mathbb{C}^{\ell+1}$ and an action of S^1 on D by rotations $\sigma \to \sigma + \alpha$. The action of $G = S^1 \times U_{\ell+1}$ lifts naturally to the action on the fields $(F, \bar{F}, \varphi, \bar{\varphi}, \psi, \bar{\psi}, \chi, \bar{\chi})$. Let Λ be an image of an element of $\mathfrak{u}_{\ell+1}$ in the representation $\mathbb{C}^{\ell+1}$. Let $\hbar$ be a generator of S^1, $v_0 = \partial_\sigma$ be a corresponding vector field on S^1 and $\mathcal{L}_{v_0}$ be the Lie derivative along v_0.

Consider G-equivariant type A topological linear sigma model on D with the target space $X = \mathbb{C}^{\ell+1}$ described by a G-invariant action functional

$$S_D = \imath \int_\Sigma d^2z \, \Big(\langle F, \bar{\partial}\varphi\rangle + \langle \bar{F}, \partial\bar{\varphi}\rangle + \langle \bar{\psi}, \partial\bar{\chi}\rangle + \langle \psi, \bar{\partial}\chi\rangle\Big), \tag{2.2}$$

The action is also invariant with respect to an odd transformation δ_G

$$\begin{aligned} &\delta_G\varphi = \chi, \qquad \delta_G\chi = -(\imath\Lambda\varphi + \hbar\,\mathcal{L}_{v_0}\varphi), \qquad \delta_G\psi = F, \qquad \delta_G F = -(\imath\Lambda\psi + \hbar\,\mathcal{L}_{v_0}\psi), \\ &\delta_G\bar{\varphi} = \bar{\chi}, \qquad \delta_G\bar{\chi} = -(-\imath\Lambda\bar{\varphi} + \hbar\,\mathcal{L}_{v_0}\bar{\varphi}), \qquad \delta_G\bar{\psi} = \bar{F}, \qquad \delta_G\bar{F} = -(-\imath\Lambda\bar{\psi} + \hbar\,\mathcal{L}_{v_0}\bar{\psi}). \end{aligned} \tag{2.3}$$

Let us remark that δ_G can be considered as an infinite-dimensional analog of the de Rham differential in the Cartan model for equivariant cohomology. Observables in the topological sigma model are given by δ_G-closed G-invariant functionals of the fields.

Theorem 2.1 *Let $V=\mathbb{C}^{\ell+1}$ be a standard representation of $U_{\ell+1}$, Λ be the image of an element $u\in\mathfrak{u}_{\ell+1}$ in $End(V)$. Then the following identity holds*

$$\Big\langle e^{\mu\,\mathcal{O}_{\Lambda,\hbar}}\Big\rangle_D\,=\,\hbar^{-\frac{\ell+1}{2}}\,\det_V\,\left(\frac{2}{\mu\hbar}\right)^{-\Lambda/\hbar}\,\Gamma(\Lambda/\hbar),\tag{2.4}$$

where $\mathcal{O}_{\Lambda,\hbar}$ is given by

$$\mathcal{O}_{\Lambda,\hbar}=\frac{\imath}{2}\int_0^{2\pi}d\sigma\,(-\langle\chi(re^{\imath\sigma}),\chi(re^{\imath\sigma})\rangle+\langle\varphi(re^{\imath\sigma}),(\imath\Lambda+\hbar\mathcal{L}_{v_0})\varphi(re^{\imath\sigma})\rangle)|_{r=1}.\tag{2.5}$$

The functional integral in the $S^1\times U_{\ell+1}$-equivariant type A topological linear sigma model (2.2) *in the l.h.s. of* (2.4) *is defined using ζ-function regularization of Gaussian integrals.*

Taking $\mu=2/\pi$, $\hbar=1$ and making the change of variables $\Lambda\to(s\,\cdot\mathrm{id}-\Lambda)/2$ the correlation function (2.4) turns into local Archimedean L-factor (1.16). Let us note that the correlation function (2.4) for arbitrary μ and $\hbar$ can be considered as an Archimedean L-factor taking into account freedom to redefine ϵ-factor in the functional equation for global L-functions.

The functional integral (2.4) can be interpreted as a $S^1\times U_{\ell+1}$-equivariant symplectic volume of the space of holomorphic maps of the disk D to $\mathbb{C}^{\ell+1}$. Let M be a $2(\ell+1)$-dimensional symplectic manifold with a symplectic form ω. Let G be a compact Lie group acting on (M,ω) and the action is Hamiltonian with the momentum map $H:\,M\to\mathfrak{g}^*$ to the dual $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ of G. Then G-equivariant symplectic volume of M is defined as an the following integral

$$Z(M,\lambda)=\int_M e^{\omega+\langle\lambda,H\rangle}=\int_M\frac{\omega^{\ell+1}}{(\ell+1)!}\;e^{\langle\lambda,H\rangle},\qquad\lambda\in\mathfrak{g},\tag{2.6}$$

where $\langle\,,\,\rangle$ is the paring between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The integral (2.6) is a finite-dimensional analog of the functional integral in the l.h.s. of (2.4) where the observable (2.5) plays the role of the equivariant symplectic form $\omega_G=\omega+\langle\lambda,H\rangle$.

3 q-version of $\mathfrak{gl}_{\ell+1}$-Whittaker function

Any local non-Archimedean factor $L_p(s)$ can be represented as a trace of Frobenius homomorphism acting in the direct sum of symmetric powers S^*V of some fixed representation V of the Galois group. Similar representation of a non-Archimedean Whittaker function as a trace of Frobenius homomorphism in finite-dimensional representations of Galois group is given in [CS]. These representations provides an arithmetic interpretation of local non-Archimedean L-factors/Whittaker functions. On the other hand Archimedean L-factors/Whittaker functions are analytic objects avoiding an analog of such interpretation. To make the corresponding structure in Archimedean case visible one can use a q-deformation of L-factors/Whittaker functions interpolating between non-Archimedean ($q=0$) and Archimedean ($q\to1$) cases. In this Section we recall a construction [GLO3] of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p}_{\ell+1})$ defined on the lattice $\underline{p}_{\ell+1}=(p_{\ell+1,1},\ldots,p_{\ell+1,\ell+1})\in\mathbb{Z}^{\ell+1}$. The q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions are common eigenfunctions of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\,=\,(\sum_{I_r}\prod_{i\in I_r}z_i)\;\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}),\tag{3.1}$$

where

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\,=\,\sum_{I_r}(X_{i_1}^{1-\delta_{i_2-i_1,1}}\cdot\ldots\cdot X_{i_{r-1}}^{1-\delta_{i_r-i_{r-1},1}}\cdot X_{i_r}^{1-\delta_{i_{r+1}-i_r,1}})T_{i_1}\cdot\ldots\cdot T_{i_r}.\tag{3.2}$$

Here the sum is over ordered subsets $I_r = \{i_1 < i_2 < \ldots < i_r\} \subset \{1, 2, \ldots, \ell+1\}$, $i_{r+1} := \ell + 2$. We use the following notations

$$T_i f(\underline{p}_{\ell+1}) = f(\underline{\widetilde{p}}_{\ell+1}), \qquad \widetilde{p}_{\ell+1,k} = p_{\ell+1,k} + \delta_{k,i}, \qquad i, k = 1, \ldots, \ell+1,$$

$$X_i = 1 - q^{p_{\ell+1,i} - p_{\ell+1,i+1} + 1}, \qquad i = 1, \ldots, \ell,$$

and $X_{\ell+1} = 1$. We also assume $q \in \mathbb{C}^*, |q| < 1$. For example, the first nontrivial Hamiltonian has the following form:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \sum_{i=1}^{\ell} (1 - q^{p_{\ell+1,i} - p_{\ell+1,i+1} + 1}) T_i \;+\; T_{\ell+1}. \tag{3.3}$$

The main result of [GLO3] is a construction of common eigenfunctions of quantum Hamiltonians (3.2) satisfying the "class one" condition (important for arithmetic interpretations [CS]). Thus one shall have

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p}_{\ell+1}) = 0, \tag{3.4}$$

outside dominant domain $p_{\ell+1,1} \geq \ldots \geq p_{\ell+1,\ell+1}$. Denote by $\mathcal{P}^{(\ell+1)} \subset \mathbb{Z}^{\ell(\ell+1)/2}$ a subset of integers $p_{n,i}$, $n = 1, \ldots, \ell+1$, $i = 1, \ldots, n$ satisfying the Gelfand-Zetlin conditions $p_{k+1,i} \geq p_{k,i} \geq p_{k+1,i+1}$ for $k = 1, \ldots, \ell$. In the following we use the standard notation $(n)_q! = (1-q)...(1-q^n)$.

Theorem 3.1 *Let $\mathcal{P}_{\ell+1,\ell}$ be a set of $\underline{p}_\ell = (p_{\ell,1}, \ldots, p_{\ell,\ell})$ satisfying the conditions $p_{\ell+1,i} \geq p_{\ell,i} \geq p_{\ell+1,i+1}$. The following recursive relation holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \sum_{\underline{p}_\ell \in \mathcal{P}_{\ell+1,\ell}} \Delta(\underline{p}_\ell)\; z_{\ell+1}^{\sum_i p_{\ell+1,i} - \sum_i p_{\ell,i}}\; Q_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_\ell | q) \Psi^{\mathfrak{gl}_\ell}_{z_1,\ldots,z_\ell}(\underline{p}_\ell),$$

where

$$\begin{gathered} Q_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_\ell | q) \;=\; \frac{1}{\prod\limits_{i=1}^{\ell} (p_{\ell+1,i} - p_{\ell,i})_q!\; (p_{\ell,i} - p_{\ell+1,i+1})_q!}, \\ \Delta(\underline{p}_\ell) = \prod_{i=1}^{\ell-1} (p_{\ell,i} - p_{\ell,i+1})_q! \;. \end{gathered} \tag{3.5}$$

The representation (3.5) is a q-analog of Givental's integral representation of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function given in Theorem 1.1 and turns into (1.8) after taking appropriate limit $q \to 1$.

Proposition 3.1 *There exists a $\mathbb{C}^* \times GL(\ell+1, \mathbb{C})$-module V such that the common eigenfunction constructed in Theorem 3.1 allows the following representation for $p_{\ell+1,1} \leq p_{\ell+1,2} \leq \ldots p_{\ell+1,\ell+1}$:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{p}_{\ell+1}) \;=\; \mathrm{Tr}_{\,V}\, q^{L_0} \textstyle\prod_{i=1}^{\ell+1} q^{\lambda H_i}, \tag{3.6}$$

Here $z_j = q^{\lambda_j}$, H_i, $i = 1, \ldots, \ell+1$ are Cartan generators of $\mathfrak{gl}_{\ell+1} = \mathrm{Lie}(GL(\ell+1, \mathbb{C}))$ and L_0 is a generator of $\mathrm{Lie}(\mathbb{C}^*)$.

Define a degenerate q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a specialization of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n, k) \;:=\; \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n+k, k, \ldots, k). \tag{3.7}$$

This degenerate q-Whittaker function is an analog of the calssical degeenrate Whittake function (1.13) and has explicit representations analogous to (1.13) and (1.15)

$$\begin{aligned}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n,k) \,&=\, \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \sum_{n_1+\ldots+n_{\ell+1}=n} \frac{z_1^{n_1}}{(n_1)_q!}\cdot\ldots\cdot\frac{z_{\ell+1}^{n_{\ell+1}}}{(n_{\ell+1})_q!}\,,\\ &= \Big(\prod_{i=1}^{\ell+1} z_i^k\Big)\oint\limits_{t=0}\frac{dt}{2\pi\imath\, t}\, t^{-n}\prod_{i=1}^{\ell+1}\Gamma_q(z_i t),\end{aligned} \tag{3.8}$$

for $n\geq 0$ and $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n,k)\,=\,0$ for $n<0$. Here we use a q-version of Γ-function

$$\Gamma_q(x)\,=\,\prod_{n=0}^{\infty}\frac{1}{1-q^n x}=\sum_{n=0}^{\infty}\frac{t^n}{(n)_q!}.$$

Simialrly to (1.15) the q-version of degenerate Whittaker function is expressed through the q-versions of a local L factor

$$L_q(s|V)=\det_V \Gamma_q(q^{s-\Lambda}), \tag{3.9}$$

where $V\,=\,\mathbb{C}^{\ell+1}$ and $\Lambda\,=\,(\Lambda_1,\ldots,\Lambda_{\ell+1})$. Thus defined L_q-factors allow a representation as a trace analogous to the representation (3.6) for Whittaker functions. The representation (3.6) can be considered as q-version of the Shintani-Casselman-Shalika formula [CS] representing non-Archimedean Whittaker function as trace of Frobenius over a finite-dimensional representation of the local Galois group. Indeed in the limit $q\to 0$ the Whittaker given in Theorem 3.1 reduces to a character of an irreducible finite-dimensional representations of $GL_{\ell+1}$ corresponding to a partition $p_{\ell+1,1}\leq\cdots\leq p_{\ell+1,\ell+1}$

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{p}_{\ell+1})\,=\,\chi^{\mathfrak{gl}_{\ell+1}}_{\underline{p}_{\ell+1}}(\underline{z}):=\sum_{p_{k,i}\in\mathcal{P}^{\ell+1}}\prod_{k=1}^{\ell+1} z_k^{(\sum_{i=1}^k p_{k,i}-\sum_{i=1}^{k-1}p_{k-1,i})}\,, \tag{3.10}$$

where we set $z_i\,=\,q^{\lambda_i}$, $i\,=\,1,\ldots,\ell+1$ and the notation $\underline{z}\,=\,(z_1,z_2,\ldots,z_{\ell+1})$ is used. Thus for $q=0$ (3.6) reproduces the non-Archimedean expression [CS]. In the next Sections we elucidate the nature of the $\mathbb{C}^*\times GL_{\ell+1}$-modules V apearing in (3.6).

4 q-Whittaker function and spaces of quasimaps

In this Section we provide an interpretation of the trace type representation (3.6) for the degenerate q-Whittaker function (3.7) and an analog of (3.6) for L_q-factors (3.9). Consider the space $\mathcal{M}_d(\mathbb{P}^\ell)$ of holomorphic maps of $\mathbb{P}^1$ to $\mathbb{P}^\ell$ of degree d. Explicitly, it can be described as a set of collections of $(\ell+1)$ relatively prime polynomials of degree d, up to a common constant factor. The space $\mathcal{M}_d(\mathbb{P}^\ell)$ allows a compactification by the space of quasi-maps $\mathcal{QM}_d(\mathbb{P}^\ell)=\mathbb{P}^{(\ell+1)(d+1)-1}$ defined as a set of collections of $(\ell+1)$ polynomials of degree d, up to a common constant factor. On the space $\mathcal{QM}_d(\mathbb{P}^\ell)$ there is a natural action of the group $\mathbb{C}^*\times GL_{\ell+1}$ (and, thus, of its maximal compact subgroup $S^1\times U_{\ell+1}$) where the action of $GL_{\ell+1}$ is induced by the standard action on $\mathbb{P}^\ell$ and the action of $\mathbb{C}^*$ is induced by the action of $\mathbb{C}^*$ on $\mathbb{P}^1$. The space of sections of the line bundle $\mathcal{O}(n)$ on $\mathcal{QM}_d(\mathbb{P}^\ell)$ is naturally a $\mathbb{C}^*\times GL_{\ell+1}$-module. Let $T\in GL_{\ell+1}$ be a Cartan torus, $H_1,\ldots,H_{\ell+1}$ be a basis in Lie(T), and L_0 be a generator of Lie$(\mathbb{C}^*)$. Let $\mathcal{L}_k$ be a one-dimensional $GL_{\ell+1}$-module such that $H_i\mathcal{L}_k=k\mathcal{L}_k$, for $i=1,\ldots,\ell+1$. Cohomology groups $H^*(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{O}(n))\otimes\mathcal{L}_k$ have a natural structure of $\mathbb{C}^*\times GL_{\ell+1}(\mathbb{C})$-module. Let $\mathcal{M}_d(\mathbb{C},\mathbb{C}^{\ell+1})$ be a space of holomorphic maps of $\mathbb{C}$ to $\mathbb{C}^{\ell+1}$ defined as a set of collections of $(\ell+1)$ polynomials of degree d and let $\mathcal{W}_d$ be a space of polynomial functions on $\mathcal{M}_d(\mathbb{C},\mathbb{C}^{\ell+1})$.

Proposition 4.1 *For the* $\mathbb{C}^*\times GL_{\ell+1}$*-character of the module* $\mathcal{V}_{n,k,d}\,=\,H^0(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{L}_k\otimes\mathcal{O}(n))$*,* $n\geq 0$ *the following integral representation holds*

$$\mathrm{Tr}_{\,\mathcal{V}_{n,k,d}}\,q^{L_0}\,e^{\sum\lambda_iH_i}\;=\;\Big(\prod_{i=1}^{\ell+1}z_i^k\Big)\oint_{t=0}\frac{dt}{2\pi\imath\,t^{n+1}}\prod_{m=1}^{\ell+1}\prod_{j=0}^{d}\frac{1}{(1-tq^jz_m)},\tag{4.1}$$

where $\underline{z}=(z_1,\ldots,z_{\ell+1})$*,* $z_m=e^{\lambda_m}$*,* H_i*,* $i=1,\ldots,\ell+1$ *are Cartan generators of* $\mathfrak{gl}_{\ell+1}=\mathrm{Lie}(GL(\ell+1,\mathbb{C}))$ *and* L_0 *is a generator of* $\mathrm{Lie}(\mathbb{C}^*)$*.*

Let us remark that the r.h.s. can be interpreted as a Riemann-Roch-Hirzebruch formula for G-equivariant holomorphic Euler characteristic of the line bundle $\mathcal{L}_k(n)=\mathcal{L}_k\otimes\mathcal{O}(n)$

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{L}_k(n))\;=\;\big\langle\mathrm{Ch}_G(\mathcal{L}_k(n))\,\mathrm{Td}_G(\mathcal{T}\mathcal{QM}_d(\mathbb{P}^\ell)),[\mathcal{QM}_d(\mathbb{P}^\ell)]\big\rangle.\tag{4.2}$$

using the standard model for equivariant K-theory of projective spaces

$$K(\mathbb{P}^N)=\mathbb{C}[t,t^{-1}]/(1-t)^{N+1},\qquad K_{U_{N+1}}(\mathbb{P}^N)=\mathbb{C}[t,t^{-1},z,z^{-1}]\Big/\prod_{j=1}^{N+1}(1-tz_j)\,.\tag{4.3}$$

Using this Proposition, q-deformed degenerate $\mathfrak{gl}_{\ell+1}$-Whittaker functions can be expressed in terms of holomorphic sections of line bundles on a space $\mathcal{L}\mathbb{P}^\ell_+$ defined as an appropriate limit of $\mathcal{QM}_d(\mathbb{P}^\ell)$ when $d\to+\infty$. Geometrically $\mathcal{L}\mathbb{P}^\ell_+$ should be considered as a space of algebraic disks in $\mathbb{P}^\ell$.

Theorem 4.1 *(i) Let* $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k)$ *be a degenerate Whittaker function* (3.7)*. Then the following holds*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k)\;=\;\lim_{d\to\infty}\,\mathrm{Tr}\,\mathcal{V}_{n,k,d}\,q^{L_0}\,e^{\sum\lambda_iH_i}\;=\;\Big(\prod_{j=1}^{\ell+1}z_j^k\Big)\oint_C\frac{dt}{2\pi\imath\,t^{n+1}}\prod_{i=1}^{\ell+1}\Gamma_q(tz_i),\tag{4.4}$$

where the integration contour C *encircles all poles except* $t=0$*.*

(ii) The following expression for a q-version of the local L-factor (3.9) *holds*

$$L_q(s|V)\;:=\;\det_V\Gamma_q(q^{s-\Lambda})=\lim_{d\to\infty}\;\mathrm{Tr}\,\mathcal{W}_d\,q^{L_0}\,q^{\sum\lambda_iH_i},\tag{4.5}$$

where $V=\mathbb{C}^{\ell+1}$ *and* $\Lambda=(\Lambda_1,\ldots,\Lambda_{\ell+1})$*,* $\Lambda_j=s-\lambda_j$*.*

Taking a limit $d\to\infty$ at the level of underlying vector spaces $\mathcal{V}_{n,k,d}$ and $\mathcal{W}_d$ can be naturally understood in terms of topological field theory interpretation of representation given in Theorem 4.1. In the following Section we provide such interpretation for q-deformed L-function (4.5).

5 Γ_q-function via equivariant linear sigma model on $D\times S^1$

In Section 2 we describe functional integral representation of a Γ-function as an equivariant symplectic volume of the space of holomorphic maps $D\to\mathbb{C}$. According to the standard Correspondence Principle in quantum/statistical mechanics such equivariant volumes provide asymptotics of the partition functions of quantum theories. Applying this reasoning to the equivariant volume considered in Section 2 and using the standard path integral interpretation of quantum mechanics we obtain the following functional representation of the q-version of Γ-function.

Theorem 5.1 *Consider a three-dimensional topological linear sigma model on $N = S^1 \times D$ with the action*

$$S = S_0 + \mathcal{O}, \tag{5.1}$$

where

$$S_0 = \imath \int_{S^1 \times D} d^2z \, d\tau \, \left(\partial_{\bar{z}}\chi\bar{\psi}_z + \bar{F}_z\partial_{\bar{z}}\varphi + \partial_z\bar{\chi}\psi_{\bar{z}} + F_{\bar{z}}\partial_z\bar{\varphi}\right), \tag{5.2}$$

and

$$\mathcal{O} = \frac{\imath}{2}\beta \int_{\partial N = S^1 \times S^1} d\tau \, d\sigma \, (\bar{\chi}\chi + \bar{\varphi}(\hbar\partial_\sigma + 2\pi\imath\beta^{-1}\partial_\tau + \imath\lambda)\varphi). \tag{5.3}$$

Then the functional integral with free boundary conditions defined using ζ-function regularization is equal to

$$Z(t,q) = \prod_{n=0}^{+\infty} \frac{1}{1 - tq^n} = \Gamma_q(t), \tag{5.4}$$

where $t = e^{-\beta\lambda}$, $q = e^{-\beta\hbar}$.

Note that similar to the two-dimensional topological theory considered in Section 2 this three-dimensional theory is also invariant with respect to odd transformations

$$\delta_{G_0}\varphi = \chi, \qquad \delta_{G_0}\chi = -(\hbar\partial_\sigma + 2\pi\imath\beta^{-1}\partial_\tau + \imath\lambda)\varphi,$$

$$\delta_{G_0}\psi_{\bar{z}} = F_{\bar{z}}, \qquad \delta_{G_0}F_{\bar{z}} = -(\hbar\partial_\sigma + 2\pi\imath\beta^{-1}\partial_\tau + \imath\lambda)\psi_{\bar{z}}.$$

Finally note the the functional integral (5.1) defined using ζ-function regularization gives a proper interpretation of the $d \to \infty$ limit considered in the previous Section.

6 Concluding remarks

The construction of the functional integral representation of local Archimedean L-factors uses an integral representation of the Γ-function. This functional integral representation should be compared with the standard Euler integral representation. One can show that the Euler integral representation naturally arises as a disk partition function in the equivariant type B topological Landau-Ginzburg model on a disk with the target space $\mathbb{C}$ and the superpotential $W(\xi) = e^{\xi} + \lambda\xi$, $\xi \in \mathbb{C}$. This result is not surprising in view of a mirror symmetry between type A and type B topological sigma model. Thus we have two integral representations of Γ-function, one is in terms of an infinite-dimensional equivariant symplectic volume and another is given by a finite-dimensional complex integral. Taking into account the mirror symmetry relating the two underlying topological theories, the two integral representations should be considered on equal footing. These two integral representations of Γ-functions are similar to two different constructions (arithmetic and automorphic) of local Archimedean L-factors. The equivalence of the resulting L-factors is a manifestation of local Archimedean Langlands correspondence. The analogy between mirror symmetry and local Archimedean Langlands correspondence looks not accidental and can eventually imply that local Archimedean Langlands correspondence follows from the mirror symmetry.

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