Dualizing spheres for compact p-adic analytic groups.

Part II applications to Chromatic Homotopy theory

Agnès Beaudry
CU Boulder
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Dualizing Spheres for p-adic analytic pgroups

Part II jt with Paul Goerss
Mike Hopkins
Vesna Stojanoska

... $\mathbb{O}_n^\times$ units in max. order of central div. alg over $\mathbb{Q}_p$ ...

... comes Aut. of f.g.l.

Example 1 $\Gamma_1(x,y) = x+y+xy \in \mathbb{F}_p[[x,y]]$

$$\text{e.g. } \underset{\Gamma_1(x,x)}{\underset{\shortparallel}{[2](x)}} \in \mathbb{Z} \subseteq \mathbb{Z}_p = \mathbb{O}_1 \qquad \underset{\mathbb{G}_1}{\underset{\shortparallel}{\mathbb{O}_1^\times}} = \mathbb{Z}_p^\times$$

Example 2 $C: y^2+y=x^3$ over $\mathbb{F}_4$, $\Gamma_2(x,y) \in \mathbb{F}_4[[x,y]]$

$$Q_8 \rtimes \mathbb{F}_4^\times \simeq \mathrm{End}(C) \subseteq \mathbb{O}_2 \qquad \mathbb{O}_2 \otimes \mathbb{Q} \cong \mathbb{Q}_2\{1,i,j,k\}$$

$\circlearrowleft \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ (acting on $\mathbb{O}_2$)

$$\mathbb{G}_2 := \mathbb{O}_2 \rtimes \mathrm{Gal}$$

$$\mathbb{O}_n = \mathrm{End}_{\mathbb{F}_{p^n}}(\Gamma_n) \qquad \mathbb{G}_n := \mathbb{O}_n^\times \rtimes \mathrm{Gal}$$

$\circlearrowleft \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ (acting on $\mathbb{O}_n$)

$\uparrow$ height n f.g.l ... $[p](x) = x^{p^n} + \dots$

Universal deformation to a complete local ring.

(R, F)

$$\begin{tikzcd} G_n \arrow[loop right] & R \arrow[r, "f"] \arrow[d, "\text{mod max}"'] & B \arrow[d] \\ & \mathbb{F}_{p^n} & \mathbb{F}_{p^n} \end{tikzcd}
\qquad
\begin{tikzcd} F \arrow[r, "f^*", "\cong"'] \arrow[d] & G \arrow[d] \\ \Gamma_n & \Gamma_n \end{tikzcd}
\quad \circlearrowleft G_n$$

$G(x,y) \in B[[x,y]]$
$\downarrow$ mod max
Γ_n

$u = \dfrac{dt}{F_x(0,t)}$

invariant diff

$$MU_* \xrightarrow{F} R[u^{\pm 1}]$$

Top space

$$X \longmapsto MU_*X \otimes_{MU_*} R[u^{\pm 1}] =: (E_n)_* X$$

Generalized coh. thy ... Spectrum $E_n \circlearrowleft G_n$
by E_∞-rings

Key

$$L_{K(n)} S^0 \simeq E_n^{hG_n}$$

E_n: Morava E thy

$L_{K(n)} S^0$: $K(n)$-local sphere

$K(n)$: Morava K-thy.

Example 1 E_1 is p-complete K-theory
$\mathbb{G}_1$ Adams operations

Example 2 E_2 elliptic coh. thy
$\tilde{C}: y^2 + 3a_1 xy + (a_1 - 1)y = x^3$

Finite res. philosophy $E_n^{h\mathbb{G}_n}$ can be finitely built from E_n^{hG} for $G \subseteq \mathbb{G}_n$ finite

higher real K-theories

Example 1 $p=2$ $(\pm 1) \subseteq \mathbb{G}_1 = \mathbb{Z}_2^\times$ $\quad K^{h(\pm 1)} = KO$

$$L_{K(1)} S^0 \to KO \xrightarrow[\psi^3 - 1]{} KO.$$

Example 2 $p=2$ $\mathrm{Aut}(C) \subseteq \mathcal{O}_2$
$\quad\quad \wr\!\wr$
$GL_2(\mathbb{F}_3) \subseteq \mathbb{G}_2$

$E_2^{hGL_2(\mathbb{F}_3)} \simeq L_{K(2)} tmf.$

Dualities $D_n X = F(X, L_{K(n)} S^0)$

Problem Identify $D_n(E_n^{hG}) \in \mathrm{Pic}(E_n^{hG})$.

$$D_n(E_n^{hG}) \simeq (DE_n)^{hG}$$

Tate vanishing
ambidexterity

Thm $D_n E_n \simeq I_{\mathbb{G}_n}^{-1} \wedge E_n$ $\quad$ top. realization Serre's dualizing sheaf.

lin. hyp. $\quad \simeq S^{-\mathfrak{g}_n} \wedge E_n$ $\quad$ p-adic Lie alg. of $\mathbb{G}_n$.

$\mathfrak{g}_n \cong \mathcal{O}_n \circlearrowleft$ conj. act. of $\mathbb{G}_n \supseteq \mathcal{O}_n^\times$

$$D_n(E_n^{hG}) \simeq (D_n E_n)^{hG} \simeq (S^{-\mathfrak{g}_n} \wedge E_n)^{hG}$$

find real model.

Example 1 $\quad \mathfrak{g}_1 = \mathbb{Z}_p \circlearrowleft \mathbb{Z}_p^\times$ act. conj.... trivial

$$\mathbb{Z} \subseteq \mathbb{Z}_p \qquad S^{\mathfrak{g}_1} \simeq S^{\mathbb{R}\otimes\mathbb{Z}} \simeq S^1$$

$$D_1 E_1 \underset{G_1}{\simeq} \Sigma^{-1} E_1 \quad \dots \text{ e.g. } D_1 KO \simeq \Sigma^{-1} KO$$

Example 2 $\quad \mathfrak{g}_2 = \mathcal{O}_2 \circlearrowleft$ conj.

$$G = Q_8 \rtimes \mathbb{F}_4^\times \qquad \mathbb{H}^{ad} = \mathbb{R}\{1,i,j,k\} \overset{G}{\circlearrowleft} \text{conj}$$

$$S^{\mathbb{H}^{ad}} \underset{G}{\simeq} S^{\mathfrak{g}_2}$$

$$D(E_2^{hG}) \simeq (S^{-\mathbb{H}^{ad}} \wedge E)^{hG} \simeq ?$$

Thm (Bobkova) $D_2 E_2^{hG} \simeq \Sigma^{44} E_2^{hG} \in Pic(E^{hG}) \cong \mathbb{Z}/192$

$$\begin{tikzcd}
RO(G) \arrow[rr, "J"] \arrow[d] & & Pic(E^{hG}) \arrow[d, "\cong"] \\
\pi_0 ko^{hG} \arrow[r] & \pi_0 pic(S^0)^{hG} \arrow[r] & \pi_0 pic(E)^{hG}
\end{tikzcd}$$

$$V \longmapsto \cdots \longmapsto (E \wedge S^V)^{hG}$$

Goal understand J

Tool HFSS $\quad ko^{hG} \to pic(E)^{hG}$

Fact $\langle \pm 1 \rangle \leq G \Rightarrow (E \wedge S^{\rho_G})^{hG} \simeq E^{hG}$.

i.e. $\rho_G \in \ker J$.

Example 1

$$\begin{tikzcd}
ko^{hG} \arrow[r] \arrow[d] & \pi_0 \mathrm{pic}(E)^{hG} \cong \mathbb{Z}/192(\Sigma E^{hG}) \\
\pi_0 ko_{\leq 7}^{hG} \arrow[ur, dashed] & \\
\mathbb{Z}\oplus\mathbb{Z}/8 \arrow[u, phantom, "\text{\rotatebox{90}{$=$}}"] \arrow[r] & \underline{\mathbb{Z}\oplus\mathbb{Z}/8} \arrow[uu, "\cong"']
\end{tikzcd}$$

String or. of TMF.

$(24, 1)$

$(\dim V, \lambda(V))$

$(\dim p_G, \lambda(p_G))$

$\lambda \in H^4(B\mathrm{Spin}, \mathbb{Z})$

$2\lambda = p_1$

Compute

$(\dim \mathbb{H}^{ad}, \lambda(\mathbb{H}^{ad}))$

$= (4, 2) \equiv (-44, 0)$

$\cong J(S^{-44})$

$\therefore\ \mathcal{D}_2 E_2^{hG} \simeq S^{44} \wedge E^{hG}.$