Motivic tensor-triangular geometry, I

① t.-classification problem

Starting point: spaces up to htpy

weaken this problem : stabilize

Recall : (X, x) pointed space

$$\pi_n(X) \xrightarrow{\Sigma} \pi_{n+1}(\Sigma X) \xrightarrow{\Sigma} \pi_{n+2}(\Sigma^2 X) \to \cdots$$

stabilizes to $\pi_n^s(X, x)$

Def^n Category SH^ω

obj (X, n) X pt finite CW, $n \in \mathbb{Z}$

maps $\hom((T,m),(X,n)) = \underset{k \to \infty}{\operatorname{colim}} \{ \text{htpy classes } \Sigma^{k+m} T \to \Sigma^{k+n} X \}$

Exa $\hom((S^0, m), (X, n)) = \pi^s_{m-n}(X)$ $\quad m-n \geq 0$

Weakened problem:

classify objects in SH^ω up to isomorphism

Exa $X \xrightarrow{f} Y \to Z$ cofiber sequence $\rightsquigarrow$

$$\cdots \to H^k(Z) \to H^k(Y) \to H^k(X) \to H^{k+1}(Z) \to \cdots$$

l.e.s.

f H^*-iso $\iff$ $H^*(Z) = 0$

Rmk $\{ Z \in SH^\omega \mid \tilde{H}^*(Z) = 0 \}$ forms thick (triangulated) subcategory

- SH^ω is a triangulated category:
  - $\Sigma : SH^\omega \xrightarrow{\sim} SH^\omega$, $(X, n) \mapsto (X, n+1)$
  - "distinguished triangles" $(X,n) \xrightarrow{(f,n)} (Y,n) \to (C(f), n) \to \Sigma(X_n)$
- T Δ-cat. $K \overset{\text{full}}{\subseteq} T$ is thick if:
  - $0 \in K$
  - $X \in K \iff \Sigma X \in K$
  - $X \to Y \to Z \to \Sigma X$ distinguished Δ: if 2 out of $\{X, Y, Z\}$ belong to K so does the 3rd
  - $X \oplus Y \in K \Rightarrow X, Y \in K$

Not^n T Δ-cat, $x, y \in T$ $\quad x \overset{t}{\sim} y \;:\!\iff \text{thick}(x) = \text{thick}(y)$

Final weakening:

Classify objects in SH^ω up to $\overset{t}{\sim}$

Theorem (Hopkins-Smith 80's/90's)

$$\{ \text{thick subcategories of } SH^\omega_{(p)} \} \longleftrightarrow \mathbb{N} \cup \{\infty\}$$
$$\ker(K(n)) \longleftarrow\!\shortmid\; n$$
$$0 \longleftarrow\!\shortmid\; \infty$$

$K(0) = H^*(-; \mathbb{Q})$

$K(\infty) = H^*(-; \mathbb{F}_p)$

② t.t.-classification problem

(1) R commutative, noetherian ring

$D(R)^\omega$

obj : $0 \to X_n \to X_{n-1} \to \cdots \to X_m \to 0$ (f.g. proj. R-modules)

maps : chain htpy classes of chain maps

- additive
- $\Sigma = [1]$
- $X_\bullet \xrightarrow{f} Y_\bullet \to C(f) \to \Sigma X_\bullet$

$\Big\}$ $D(R)^\omega$ Δ-cat

Theorem (Hopkins)

$$\{ \text{thick subcats} \} \longleftrightarrow \{ \text{spc subsets of } \operatorname{Spec}(R) \}$$
$$X_\bullet \longmapsto \operatorname{supph}(X_\bullet)$$
$$K \longmapsto \bigcup_{X_\bullet \in K} \operatorname{supph}(X_\bullet)$$

(2) X algebraic variety

$D(X)^\omega$

Theorem (Thomason 90's)

$$\{ \text{thick } \underbrace{\text{tensor ideals}}_{K \otimes D(X)^\omega \subseteq K} \} \longleftrightarrow \{ \text{spc subsets of } X \}$$

Not^n $x, y \in T$, T $\otimes$-Δ-cat

$x \overset{t.t.}{\sim} y \iff \langle x \rangle = \langle y \rangle$

thick $\otimes$-ideal generated by x

Rmk
- $\underset{\wedge}{SH^\omega}$, $\underset{\otimes_R}{D(R)^\omega}$ $\otimes$-Δ-cats

  $x \overset{t}{\sim} y \iff x \overset{t.t.}{\sim} y$
- often invariants (weak) Künneth ($\Rightarrow$ ker t.t.-ideals)

Summary T $\otimes$-Δ-cat

t.t.-classification problem : classify t.t.-ideals in T.