Functors between Fukaya categories.

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§0: HMS in the compact Fano case.

- Auroux, "Mirror symmetry and T-duality in the complement of an anticanonical divisor".

§1: the partially-wrapped Fukaya category.

§2: Functors from stop-removal and open inclusion.

- Ganatra - Pardon - Shende, "Sectorial descent for wrapped Fukaya categories". ("GPS 2")

§1. Recall two foundational conjectures:

Homological mirror symmetry conjecture (Kontsevich, '94).

For certain pairs of X - symplectic mfd w/ $c_1(TX) = 0$,
$\check{X}$ - "mirror" $\mathbb{C}$-variety,

$$D^{\pi} \mathrm{Fuk}(X) \simeq D^b \mathrm{Coh}(\check{X}).$$

△

Strominger - Yau - Zaslow conjecture ('96).

$\check{X}$ can be constructed like so:

- choose a fibration of X by special Lagrangian tori
- dualize these tori to form $\check{X}$.

say Ω a hol.-volume form on X.
L is sLag if $\mathrm{im}(\Omega|_L) = 0$.

△

Call a Kähler mfld (X, ω, J) **Fano** if K_X^* is ample

$\Rightarrow$ can choose hol. volume form Ω on $X \setminus D$.

(eg. $\mathbb{CP}^n$, deg-d hypersurface in $\mathbb{CP}^n$ w/ $d \le n$)

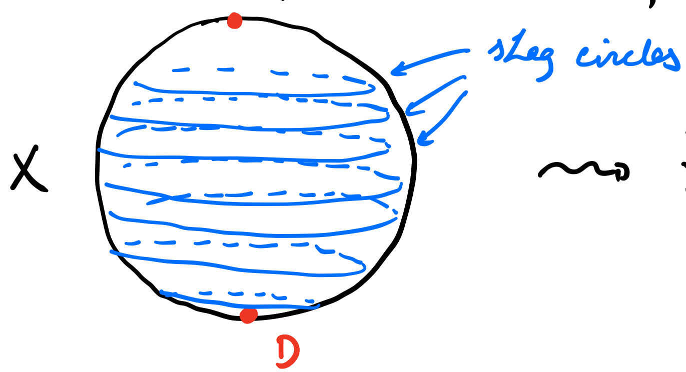
Conjecture (Hori-Vafa, '00): For X Fano, can produce a mirror $(\check{X}, W)$ like so:

- $\check{X} :=$ moduli of sLag tori in $X \setminus D$ w/ flat $U(1)$-connections
- $W : \check{X} \to \mathbb{C}$ is a hol. function defined by counting disks:

$$W(L, \nabla) := \sum_{\substack{\beta \text{ a rigid disk} \\ \text{w/ } \partial\beta \subset L}} n_\beta(L) \underbrace{e^{-\int_\beta \omega} \, \mathrm{hol}_\nabla(\partial\beta)}_{\substack{\shortparallel \\ z_\beta}} .$$

Δ

Ex. $X := \mathbb{CP}^1$, $D := \{0, \infty\}$, $\Omega := \frac{dz}{z}$.



$\rightsquigarrow \check{X} = \mathbb{C}^*$.

$z_\beta z_{\beta'} = e^{-\int_{\mathbb{CP}^1} \omega}$;

in fact, $W = z + \dfrac{e^{-\int_{\mathbb{CP}^1} \omega}}{z}$.

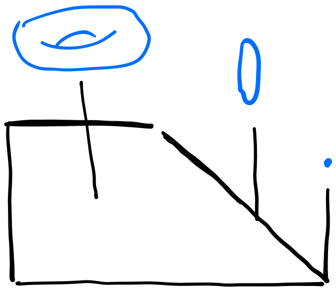
Δ

Ex. More generally, say X - smooth toric (eg. $\mathbb{CP}^n$, Hirzebruch surfaces).

$$\rightsquigarrow \quad \begin{array}{c} X^{2n} \\ \downarrow \\ \Delta \subset \mathbb{R}^n \end{array} \quad \text{"moment polytope"} .$$

Over $\mathring{\Delta}$, fibers are Lagrangian tori; tori degenerate over $\partial\Delta$

eg. F_1:



.

$$\left\{ \begin{array}{r} x \geq 0 \\ y \geq 0 \\ -y+1 \geq 0 \\ -x-y+2 \geq 0 \end{array} \right. \rightsquigarrow \boxed{W = x + y + \frac{e^{-2\pi}}{y} + \frac{e^{-4\pi}}{xy}.}$$

Δ

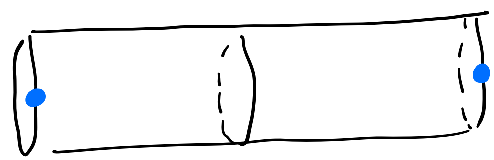
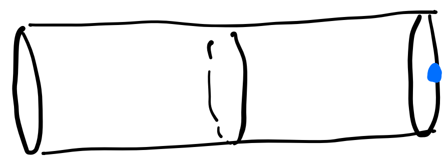
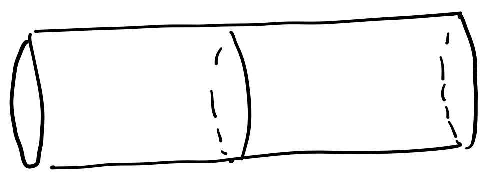
Thm (Cho-Oh): Say facets of Δ given by $\langle \nu(F), - \rangle + \alpha(F) = 0$

$$\rightsquigarrow \boxed{W = \sum_F e^{-2\pi\alpha(F)} z^{\nu(F)}.} \qquad \square$$

Expectation: $D^b\mathrm{Coh}(X) \simeq D^\pi \underbrace{\mathrm{WFuk}(\check{X}, W^{-1}(+\infty))}_{\text{"partially-wrapped Fukaya category"}}$

[Abouzaid - Seidel, Auroux, Sylvan, Ganatra - Pardon - Shende]

Ex.

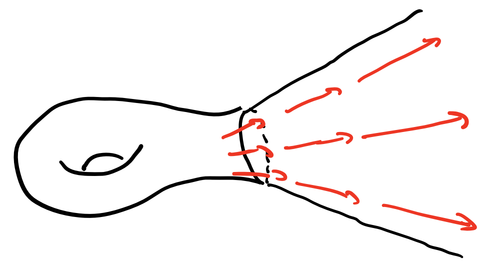
X	$(\check{X}, W^{-1}(+\infty))$		
$\mathbb{CP}^1$		$W = z + \frac{e^{-\Lambda}}{z}$	
$\mathbb{C}$		$W = z$	
$\mathbb{C}^*$		$W = 0$	

§2. Let's recall the setting for WFuk.

A **Liouville domain** is (X^{2n}, λ), w/ $\begin{cases} X \text{ - compact w/ bdry,} \\ \lambda \in \Omega^1(X),\ \underbrace{d\lambda}_{=:\,\omega} \text{ symplectic,} \end{cases}$

satisfying the property that the **Liouville v.f.** $V \in \mathfrak{X}(X)$, $\omega(V,-) = \lambda$, points outward along ∂X.

Can complete to a **Liouville manifold**: $\overbrace{(X, \lambda) \cup_{\partial X} (\partial X \times [1,\infty),\ r\lambda)}^{\hat{X}}$.



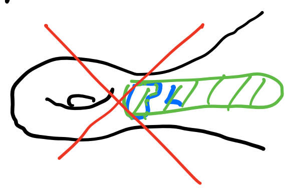
Why this defn?

(1) $X \perp$ to $\partial X \Rightarrow \lambda \wedge (d\lambda)^{n-1} = \iota_X d\lambda \wedge (d\lambda)^{n-1} = \frac{1}{n} \iota_X (d\lambda)^n$

$\Rightarrow (\lambda \wedge (d\lambda)^{n-1})|_{\partial X} \neq 0 \Rightarrow (\partial X, \lambda)$ a **contact mfd**.

(2) J-hol. curves in $\hat{X}$ satisfy a maximum principle $\Rightarrow$ a priori C^0 bounds.

$\rightsquigarrow$ exclude info escaping to ∞.



(3) And why exactness, ie. $\omega = d\lambda$?

- say $u : S^2 \to X$ J-hol.

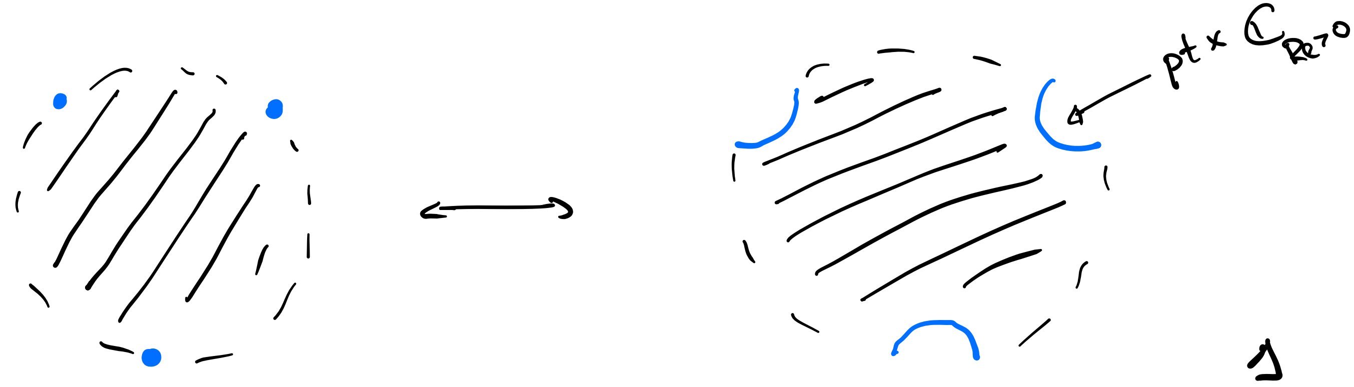
$$\int_{S^2} |du|^2 \, dvol = \int_{S^2} u^* \omega = \int_{S^2} d(u^* \lambda) = 0 \implies \text{no J-hol spheres!}$$

- if $L \subset X$ is <u>exact</u>, ie. $\lambda|_L = df$, no J-hol disks on L.

A <u>stop</u> is a closed subset $f \subset \partial X_\infty$; often consider pairs (X, f).

By removing nbhd(f), obtain a <u>Liouville sector</u> (exact symplectic mfd-w-bdry, cylindrical at ∞).

<u>Ex. ($\mathbb{C}^*$, 3 pts)</u>.

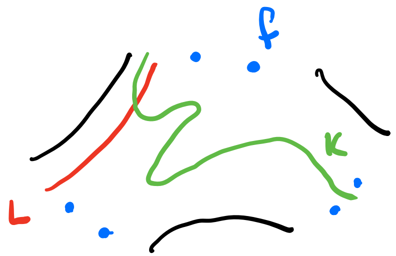


(Can also consider Liouville sectors w/ stops.)

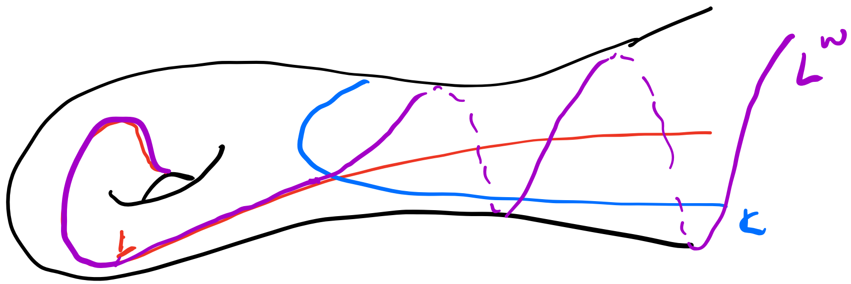
Partially-wrapped Floer cohomology.

(X, f) - stopped Liouville sector.

$L, K \subset X$ - exact, properly-embedded, cylindrical @ ∞, disjoint from ∂X, don't hit f @ ∞.

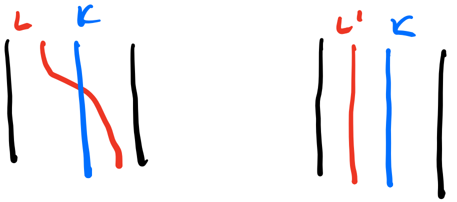


$\rightsquigarrow$ $\boxed{HF^*_f(L, K) := \lim\limits_{\substack{\longrightarrow \\ \text{positive isotopies} \\ \text{avoiding } \partial X,\, f}} HF^*(L^w, K)}$



why wrap? (1) $HF^*(L, K)$ not invariant under noncompact isotopies.

$T^*[0,1]$



$L \sim L'$, but $\operatorname{rk} HF^*(L, K) = 1$

$\operatorname{rk} HF^*(L', K) = 0$.

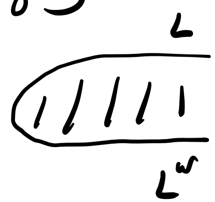
(2) need something ∞ for HMS: $\mathbb{C}^* \longleftrightarrow T^*S^1$

$\hom(\mathcal{O}, \mathcal{O}) = \mathbb{C}[z, z^{-1}]$

$\mathrm{WFuk}(X, f)$.

A_∞ - category w/ following properties:

- objects are $L \subset X$ exact, properly-embedded, cylindrical @ ∞, disjoint from ∂X, don't hit f @ ∞.
- $H^* \hom(L, K) = HF^*(L^w, K)$.
- L, L^w are isomorphic.

$\mathrm{WFuk}(X, f)$ is the universal category obtained from $\mathrm{Fuk}(X, f)$ by localizing w/ <u>continuation elements</u> 

$\rightsquigarrow$ L, L^w isomorphic.

Stop removal.

X - Liouville sector, $\mathfrak{f} \subset \mathfrak{f}' \subset \partial X_\infty$ stops.

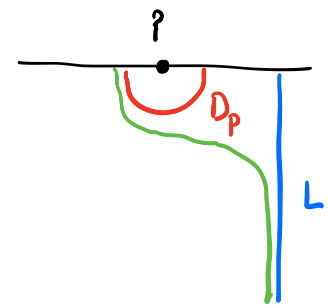
$\rightsquigarrow$ get **stop-removal functor** $\mathcal{W}\mathrm{Fuk}(X, \mathfrak{f} \cup \Lambda) \longrightarrow \mathcal{W}\mathrm{Fuk}(X, \mathfrak{f})$

(just "wrap more"!)

- identity on objects.
- $$\begin{array}{ccc} HF^*_{X,\mathfrak{f}'}(L,K) & & HF^*_{X,\mathfrak{f}}(L,K) \\ \| & & \| \\ \displaystyle\lim_{\substack{L \rightsquigarrow L^w, \\ \text{avoiding } \mathfrak{f}'}} HF^*(L^w,K) & \xrightarrow[\substack{\text{inclusion of} \\ \text{subsystem} \\ (\text{"wrap more"})}]{} & \displaystyle\lim_{\substack{L \rightsquigarrow L^w, \\ \text{avoiding } \mathfrak{f}}} HF^*(L^w,K) \end{array}$$

The wrapping exact triangle and a characterization of stop removal.

$$\left.\begin{array}{l} (X, f) \text{ - stopped Liouville sector} \\ p \in X, \quad D_p \subset X \text{ its linking disk} \end{array}\right\} \rightsquigarrow \text{exact triangle} \quad \begin{tikzcd} L^w \arrow[rr] & & L \arrow[dl] \\ & D_p \arrow[ul] & \end{tikzcd} \quad (*)$$

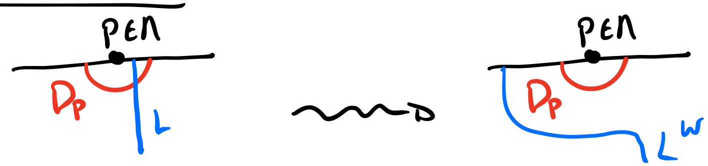


<u>Cor.</u> X - Liouville sector, $f \subset \partial X_\infty$ stop, $\Lambda \subset \partial X_\infty$ Legendrian.

$\rightsquigarrow$ stop-removal descends to $\mathrm{WFuk}(X, f \cup \Lambda)/_{\mathcal{D}} \xrightarrow{\simeq} \mathrm{WFuk}(X, f)$,

$\mathcal{D}$:= full subcategory generated by linking disks to Λ.

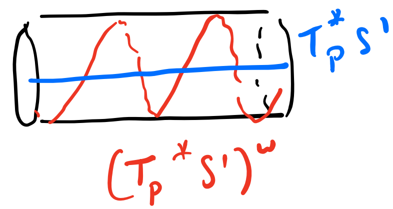
<u>Partial pf.</u> • <u>stop-removal descends</u>: $\forall L$, want $HW^*(L, D_p) = 0$ in $\mathrm{WFuk}(X, f)$.



• full-faithfulness follows from $(*)$. $\square$

Example: computing $\mathrm{WFuk}(T^*S^1)$.

$D^\pi \mathrm{WFuk}(T^*S^1) \simeq \mathrm{Perf}(\mathbb{Z}[t,t^{-1}])$. (Idea: generated by fiber $T_p^*S^1$;

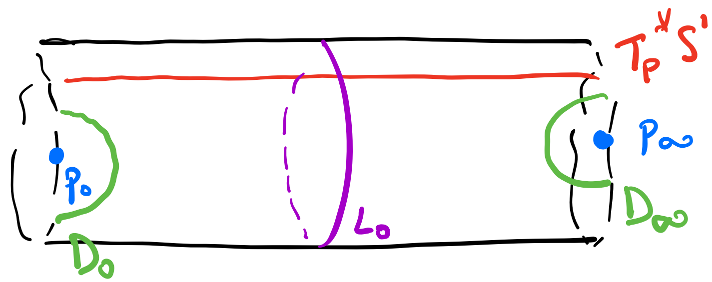


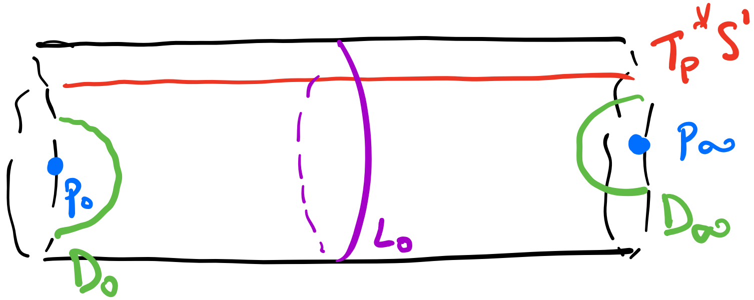
$\rightsquigarrow HF^*(T_p^*S^1, T_p^*S^1) = \mathbb{C}[t,t^{-1}]$).

But what if we knew HMS for $\mathbb{C}^*$, and wanted to use this to compute $D^\pi \mathrm{WFuk}(T^*S^1)$?

Use stop removal!

$$\mathbb{CP}^1 \longleftrightarrow (T^*S^1, 2\text{ pts})$$
$$\mathcal{O}_0, \mathcal{O}_\infty \longleftrightarrow D_0, D_\infty$$
$$\mathcal{O}_{\mathbb{CP}^1} \longleftrightarrow T_p^*S^1$$
$$\mathcal{O}_p \longleftrightarrow L_0$$





$$\mathrm{WFuk}(T^*S^1) = \mathrm{WFuk}(T^*S^1, p_0 \cup p_\infty) / \langle D_0, D_\infty \rangle$$

$$= \mathrm{Coh}(\mathbb{CP}^1) / \langle \mathcal{O}_0, \mathcal{O}_\infty \rangle$$

$$= \mathrm{Coh}(\mathbb{CP}^1 \setminus \{0, \infty\})$$

$$= \mathrm{Coh}(\mathrm{Spec}\, \mathbb{C}[t, t^{-1}])$$

$$= \mathrm{Perf}\ \mathbb{C}[t, t^{-1}] .$$

$\triangle$

(prototypical!)

The inclusion functor.

An <u>inclusion</u> of <u>Liouville sectors</u> $i : X \hookrightarrow X'$ is a proper map w/ $i^*\lambda' = \lambda + df$, supp f compact, $i : X \xrightarrow{\cong} i(X)$.

<u>Thm (GPS1)</u>: $i : X \hookrightarrow X'$ induces $\mathcal{W}\mathrm{Fuk}(X) \longrightarrow \mathcal{W}\mathrm{Fuk}(X')$. $\square$

idea:

- $L \subset X \longmapsto L \subset X \hookrightarrow X'$.

- $HF^*_X(L,K) = HF^*_{X',\,\text{wrap in } X}(L,K) \longrightarrow HF^*_{X'}(L,K)$.

<u>Cor</u>: For $F^{2n-2} \subset \partial X_\infty$ Liouville, $\mathcal{W}(F) \hookrightarrow \mathcal{W}(X, \mathfrak{f} \sqcup \mathfrak{f}_+)$. $\square$

The descent formula, and computing T^*Q for Q a mfld-w-bdry.

Sectorial descent. Say X - Liouville sector, $X = \bigcup_{1 \le i \le n} X_i$ a Weinstein sectorial covering.

Then $\exists$ $\boxed{\operatorname{hocolim}_{\emptyset \neq I \subset \{1,\dots,n\}} \mathrm{WFuk}\left(\bigcap_I X_i\right) \xrightarrow{\simeq} \mathrm{WFuk}(X).}$

(Refines Kontsevich's conjecture: WFuk is global sections of a cosheaf of categories defined over the skeleton)

Ex. X - punctured surface, Γ a ribbon graph as its core $\rightsquigarrow$ sectorial covering by "A_{n-1} Liouville sectors". Δ

Ex. Fix Q - compact manifold-w/-bdry. What's $\mathrm{WFuk}(T^*Q)$?

- [Abouzaid]: generated by T_p^*Q.
- $CF^*(T_p^*Q, T_p^*Q)$? Choose $Q = \bigcup_{1 \le i \le n} Q_i$, s.th. all intersections are balls.

$$(*) \Longrightarrow CF^*(T_pQ, T_pQ) \simeq C_{-*}(\Omega_p Q) \text{ as } A_\infty\text{-algebras.}$$

$$\left(\text{Eg. } HF^*(T_pS^1, T_pS^1) \simeq H_{-*}(\Omega_p S^1) \simeq \mathbb{C}[t, t^{-1}].\right) \quad \Delta$$