Manifolds of Positive Curvature

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We discussed recent joint work with Karsten Grove and Luigi Verdiani in which we construct a new example with positive sectional curvature in dimension 7:

THEOREM A. *There exists a seven dimensional manifold P with positive sectional curvature with the following properties:*

(a) *P is homeomorphic to the unit tangent bundle of $\mathbb{S}^4$.*
(b) $\mathrm{SO}(4)$ *acts isometrically on P with one dimensional quotient (a so called cohomogeneity one manifold).*
(c) *There exists an orbifold principal fibration* $\mathrm{SU}(2) \to P \to \mathbb{S}^4$, *where* $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ *acts almost freely on P.*

We do not know wether the manifold P is diffeomorphic to the unit tangent bundle or not.

The orbifold structure on the base is as follows: The metric is smooth, except along a standard Veronese embedding $\mathbb{RP}^2 \subset \mathbb{S}^4$, where normal to the surface the metric has an angle $2\pi/3$. The quotient is thus homeomorphic to $\mathbb{S}^4$.

The positively curved metric is a Kaluza Klein metric (sometimes called connection metric) in the orbifold principal fibration in (c). It is thus described by a metric on the base and a principal connection. Due to (b) it is sufficient to describe the metric along a geodesic orthogonal to all orbits. Along this geodesic our metric and principal connection is given by piecewise polynomial functions of degree at most 5.

The proof that the metric has positive curvature is obtained by using Thorpe's method. Here one modifies the curvature operator $\hat{R}\colon \Lambda^2 T \to \Lambda^2 T$ with a curvature type endomorphism $\hat{\alpha}\colon \Lambda^2 T \to \Lambda^2 T$ induces by a 4-form $\alpha \in \Lambda^4 T$. If the modified curvature operator $\hat{R} + \hat{\alpha}$ is positive definite, the sectional curvature is positive. We construct an explicit 4-form consisting of piecewise rational functions and combine Sylvester's theorem and Sturm's theorem to show that the minor determinants are all positive.

The example fits into an infinite family of "candidates" coming from the following classification theorem:

THEOREM B (Verdiani, n even, Grove-Wilking-Ziller, n odd). *Let M be a positively curved compact simply connected manifold on which G acts isometrically with* $\dim M/G = 1$. *Then M^n is equivariantly diffeomorphic to one of the following:*

(a) *A rank one symmetric space with a linear action of G.*
(b) *One of the normal homogeneous manifolds of positive curvature, or certain positively curved Eschenburg or Bazaikin spaces which admit a cohomogeneity one action.*
(c) *One of the seven dimensional manifolds P^7_k, Q^7_k , $k \geq 1$, or R^7 with* $G = \mathrm{SO}(4)$.

The manifolds in part (c) are not yet known to admit positive curvature, although $P_1 = \mathbb{S}^7$ and $Q_1 = \mathrm{SU}(3)/\mathrm{S}^1$ (an Aloff-Wallach space) do. Our new example is the manifold P_2.

Two theorem's by Grove-Ziller imply that all candidates in (c) admit a G invariant metric with non-negative sectional curvature and one of positive Ricci curvature as well.

The manifolds P_k are 2-connected with $\pi_3(P_k) = \mathbb{Z}_k$, and thus rational homology spheres. A finiteness theorem due to Petrunin-Tuschmann and Fang-Rong implies that the pinching constants δ_k, i.e. $0 < \delta_k \leq \sec \leq 1$, for any positively curved metric on P_k would necessarily go to 0 as $k \to \infty$, and P_k would be the first examples of this type.

For the manifolds in part (c), $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ acts almost freely and they are thus all the total space of principal orbifold bundles over $\mathbb{S}^4$ for P_k and over $\mathbb{CP}^2$ for Q_k and R. The total space in case of P_k and Q_k admit so called 3-Sasakian metrics, which already have lots of positive curvature by definition, and the induced metric on the base is the self dual Einstein orbifold metric constructed by Hitchin. The bundle can also be described, up to a 2-fold cover, as the frame bundle of the vector bundle of self dual 2-forms on the base. Our metric on $P = P_2$ is a deformation of the 3-Sasakian metric.