Introduction to superselection sector theory II

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The story so far...

We have considered the toric code on the $\mathbb{Z}^2$-lattice and constructed:

- A pure frustration-free ground state ω_0
- Automorphisms ρ^k describing single anyons (electric, magnetic and electromagnetic)
- This gives four equivalence classes of irreducible representations

Questions

- How do we know how to choose which representations?
- Does this set of representations have more structure?

The superselection criterion

Superselection rules

Consider representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ with $\psi_1, \psi_2 \in \mathcal{H}$ unit vectors. Let $\psi_\theta = \frac{\psi_1 + e^{i\theta}\psi_2}{\sqrt{2}}$ and define the state

$$\omega_\theta(A) := \langle \psi_\theta, \pi(A)\psi_\theta \rangle.$$

The (state) vectors ψ_1, ψ_2 satisfy a superselection rule[1] if the expectation values are independent of the relative phase! (This can only happen if π is *not* irreducible!)

Definition

Two states ω_1, ω_2 are called *not superposable* if in any representation π that contains vectors ψ_1, ψ_2 implementing the states, we have that

$$\omega_{\alpha\psi_1+\beta\psi_2}(A) = |\alpha|^2\omega_{\psi_1}(A) + |\beta|^2\omega_{\psi_2}(A) = |\alpha|^2\omega_1(A) + |\beta|^2\omega_2(A)$$

for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

[1] Wick, Wightman and Wigner, Physical Review, 88:101–105, 1952

Superposable irreducible representations

Theorem

Let ω_1, ω_2 be pure states of $\mathfrak{A}$. Then they are superposable iff their GNS representations π_{ω_1} and π_{ω_2} are unitarily equivalent.

- Equivalent representations have the same (normal) states
- Can think of different equivalence classes as describing different 'charges'
- Total charge cannot be changed with (quasi-)local operators!
- The vectors in the representation can describe many excitations (but all have the same 'total charge')

Problem

There are many 'unphysical' irreps of $\mathfrak{A}$!

GNS representations for anyon states

Recall that $(\pi_0, \Omega, \mathcal{H}_0)$ is the GNS representation of the frustration free ground states. Since the maps ρ_x^k are automorphisms of $\mathfrak{A}$, $\pi_0 \circ \rho_x^k$ is again a representation. Moreover, Ω is cyclic for this representation. We have

$$\langle \Omega, \pi_0 \circ \rho_x^k(a)\Omega \rangle = \omega_0(\rho_x^k(a)).$$

Now let ρ_x and ρ_x' be two such automorphisms defined in terms of semi-infinite ribbons ξ and ξ' with the same endpoint. Then $\omega_0 \circ \rho_x = \omega_0 \circ \rho_x'$, so by uniqueness of the GNS representation there must be a unitary $V \in \mathfrak{B}(\mathcal{H}_0)$ such that

$$V\pi_0 \circ \rho_x(a) = \pi_0 \circ \rho_x'(a)V.$$

These are called charge transporters.

The superselection criterion

Definition (Superselection criterion)

Let π_0 be an irreducible "reference" representation of $\mathfrak{A}$. Then π satisfies the superseleciton criterion if

$$\pi \upharpoonright \mathfrak{A}(\Lambda^c) \cong \pi_0 \upharpoonright \mathfrak{A}(\Lambda^c)$$

for all cones Λ.

- Interpretation is that of localisable and transportable representations.
- An equivalence class is called a (superselection) sector
- A general C^*-algebra has many inequivalent representations, but for a given π_0, not many sectors!
- Choice of cone depends on class of models to study.

Sectors of the toric code

Theorem

There are (at least) four irreducible sectors for the toric code.

Proof.

Fix a cone Λ. Choose a semi-infinite path ξ_k for each $k = X, Y, Z$ inside the cone. Then $\pi_0 \circ \rho^k_{\xi_k}(a) = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda^c)$. Let Λ' be a different cone, and choose paths $\xi'_k \subset \Lambda'$ as above. Then by independence of the state $\omega_0 \circ \rho^k_{\xi_k}$ on the path (plus moving a charge over a finite distance), it follows that $\pi_0 \circ \rho^k_{\xi_k} \cong \pi_0 \circ \rho^k_{\xi'_k}$. Moreover, from the previous results the four representations $\pi_0 \circ \rho^k_{\xi_k}$ are all inequivalent, and hence in distinct sectors. □

Remark

It turns out these are all irreducible sectors, but we will come back to this later.

What's next

We considered the toric code on the $\mathbb{Z}^2$ lattice:

- Constructed four types of automorphisms ρ_x^k $(k = 0, X, Y, Z)$
- The representations satisfy the superselection criterion:

$$\pi_0 \circ \rho_x^k \upharpoonright \mathfrak{A}(\Lambda^c) \cong \pi_0 \upharpoonright \mathfrak{A}(\Lambda^c)$$

  for all cones Λ
- Representations have the interpretation of describing an anyon
- Anyons are localizable and transportable

We can define extra structure on this set of representations, such as fusion and braiding!

Monoidal/tensor categories

Definition

A *monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ together with a distinguished object $1_{\mathcal{C}} \in \mathcal{C}$ and the following families of natural isomorphisms:

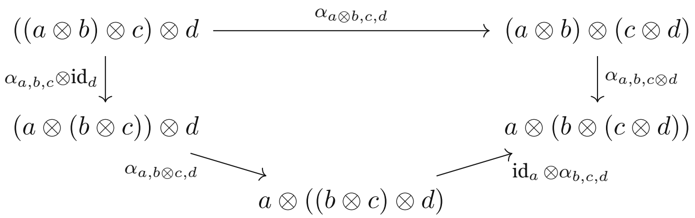
1. Associators $\alpha_{a,b,c} : (a \otimes b) \otimes c \xrightarrow{\simeq} a \otimes (b \otimes c)$
2. Unitors $\lambda_a : 1_{\mathcal{C}} \otimes a \xrightarrow{\simeq} a$ and $\rho_a : a \otimes 1_{\mathcal{C}} \xrightarrow{\simeq} a$

for all $a, b, c \in \mathcal{C}$. These should satisfy the *pentagon* and *triangle* axioms.

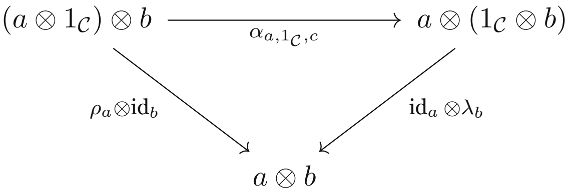
Definition

If the associators and unitors are the identity, we say that $\mathcal{C}$ is a strict monoidal category.

Pentagon axiom



Triangle axiom



A warm-up

Example

Let $\mathfrak{A}$ be a unital C^*-algebra. Then we can define the category $\mathrm{End}(\mathfrak{A})$ of unital *-endomorphisms of $\mathfrak{A}$, with the following morphisms:

$$\mathbf{Hom}_{\mathrm{End}(\mathfrak{A})}(\rho, \sigma) := \{T \in \mathfrak{A} : T\rho(a) = \sigma(a)T \quad \forall a \in \mathfrak{A}\},$$

with composition the composition of morphisms.
This has a $\otimes$-product, defined objects as $\rho \otimes \sigma := \rho \circ \sigma$. If $S \in \mathbf{Hom}(\rho_1, \rho_2)$ and $T \in \mathbf{Hom}(\sigma_1, \sigma_2)$, define

$$S \otimes T := S\rho_1(T) \in \mathbf{Hom}(\rho_1 \otimes \sigma_1, \rho_2 \otimes \sigma_2).$$

This makes $\mathrm{End}(\mathfrak{A})$ into a strict monoidal category. (Exercise!)

Charge transporters

The category $\Delta_{\mathbf{DHR}}$

Motivated by the example $\mathbf{End}(\mathfrak{A})$, we define:

Definition (DHR category, first attempt)

Given a representation π_0, the category $\Delta_{\mathbf{DHR}}$ has as objects endomorphisms ρ of $\mathfrak{A}$ which are

- localised, i.e. there is some cone Λ such that $\rho(a) = a$ for all $a \in \mathfrak{A}(\Lambda^c)$;
- transportable, i.e. for any other cone Λ', there is a ρ' localised in Λ' and a unitary $v \in \mathfrak{B}(\mathcal{H}_0)$ such that

$$v\pi_0(\rho(a)) = \pi_0(\rho'(a))v.$$

The morphisms are the intertwiners, i.e.

$$(\rho, \sigma) := \{s \in \mathfrak{B}(\mathcal{H}_0) : s\pi_0 \circ \rho(a) = \pi_0 \circ \sigma(a)s \quad \forall a \in \mathfrak{A}\}.$$

Note: we have $v \in (\rho, \rho')$ for the charge transporters.

Some remarks on Δ_{DHR}

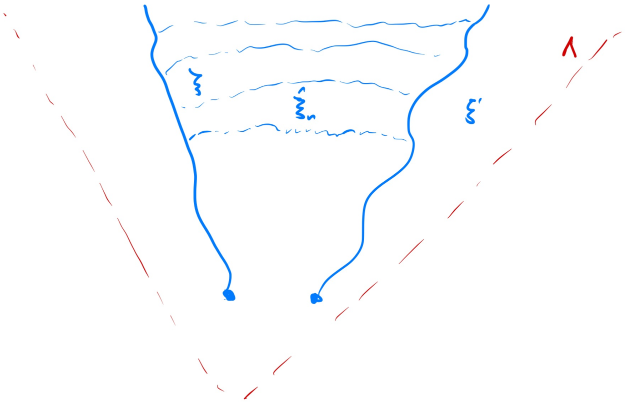
- The automorphisms ρ_x^k defined earlier are in Δ_{DHR}
- Conversely, if $\rho \in \Delta_{\mathrm{DHR}}$, $\pi_0 \circ \rho$ satisfies the superselection criterion
- The category depends on the choice of π_0 via the transportability condition!
- This is not a subcategory of $\mathbf{End}(\mathfrak{A})$ since the morphisms (charge transporters) in Δ_{DHR} need not be in $\pi_0(\mathfrak{A})$
- In particular, monoidal product will be more complicated

Charge transporters

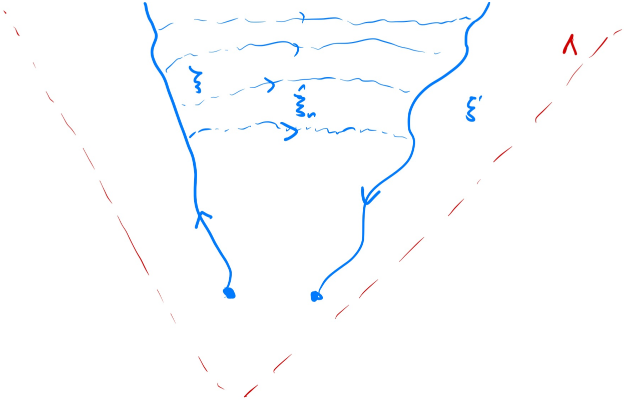
We can explicitly construct charge transporters v:

- Consider semi-finite ribbons $\xi, \xi' \subset \Lambda$ with the same endpoint x and look at the corresponding automorphisms ρ and ρ'
- Write ξ_n (ξ'_n) for the first n edges on the path.
- For each n, choose path $\widehat{\xi}_n$ connecting ends of ξ_n and ξ'_n ...
- ...such that $\mathbf{dist}(\widehat{\xi}_n, x) \to \infty$
- Define $v_n := F_{\xi_n}$. Then $\mathbf{lim}_{n\to\infty} v_n \rho(a) - \rho'(a) v_n = 0$ for all $a \in \mathfrak{A}$.
- Interpretation: v_n moves back the excitation along ξ, then go back along ξ' via $\widehat{\xi}_n$

Charge transporters



Charge transporters



Interlude: von Neumann algebras

Definition

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ be a unital $*$-subalgebra. Then $\mathcal{M}$ is called a von Neumann algebra if $\mathcal{M} = \mathcal{M}''$.

Theorem (Bicommutant theorem)

The following are equivalent:

1. *$\mathcal{M}$ is a von Neumann algebra*
2. *$\mathcal{M}$ is closed in the weak operator topology:*
   $x_\lambda \to x$ *wot* $\Leftrightarrow \langle \phi, (x_\lambda - x)\psi \rangle \to 0$ *for all* $\phi, \psi \in \mathcal{H}$.
3. *$\mathcal{M}$ is closed in the strong operator topology:*
   $x_\lambda \to x$ *sot* $\Leftrightarrow \|(x_\lambda - x)\psi\| \to 0$ *for all* $\psi \in \mathcal{H}$.

Charge transporters

Warning

The sequence v_n does not converge (in norm) to an element in $\mathfrak{A}$ in general.

However, $\pi_0(v_n)$ does converge (to a unitary) in the strong operator topology.

Proof (sketch).

It is enough to show that $v_n a \Omega$ is a Cauchy sequence for $a \in \mathfrak{A}_{\mathbf{loc}}$. Note that for n large enough, $\mathbf{supp}\, a \cap \mathbf{supp}(v_n)$ will be constant. For such n, decompose v_n as product of three path operators, such that the middle part has empty intersection with the support of a. Using that $F_\xi \Omega$ only depends on the endpoints of ξ, it follows that for each $a \in \mathfrak{A}_{\mathbf{loc}}$, for $n > k$ with k large enough,

$$v_n a \Omega = F_{\xi_k} F_{\widetilde{\xi}_n} F_{\xi'_k} a \Omega = F_{\xi_k} F_{\xi'_k} a F_{\widetilde{\xi}_n} \Omega$$

□

Charge transporters

Let Λ be a cone containing the localisation regions of ρ and ρ':

- It follows that $v \in \pi_0(\mathfrak{A}(\Lambda))''$ ...
- ...and in fact $v\pi_0(\rho(a)) = \pi_0(\rho'(a))v$, i.e. $v \in (\rho, \rho')$

Definition

Let Λ be a cone. Then we define the cone von Neumann algebra $\mathcal{R}_\Lambda := \pi_0(\mathfrak{A}(\Lambda))''$.

Haag duality

Definition (Haag duality)

We say a representation π_0 of $\mathfrak{A}$ satisfies Haag duality for cones if $\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$. Or in other words, $\mathcal{R}_\Lambda = \mathcal{R}'_{\Lambda^c}$.

Theorem (Fiedler-PN)

Haag duality for cones holds in all abelian quantum double models.

Remark: the direction $\pi_0(\mathfrak{A}(\Lambda))'' \subset \pi_0(\mathfrak{A}(\Lambda^c))'$ always holds by locality.

Remark

In the example of the toric code, we can construct everything explicitly and Haag duality is only necessary to show completeness.

Application I: localisation of intertwiners

Lemma

Let Λ_1 and Λ_2 be two cones both contained in a larger cone Λ, and suppose that ρ_i is localised in Λ_i. That is, $\rho_i(a) = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda_i^c)$. If $v \in (\rho_1, \rho_2)$, then $v \in \pi_0(\mathfrak{A}(\Lambda))''$.

Proof.

Consider $a \in \mathfrak{A}(\Lambda^c)$. Then we have

$$v\pi_0(a) = v\rho_1(a) = \rho_2(a)v = \pi_0(a)v,$$

where we used that the ρ_i are localised in Λ twice. But this implies $v \in \pi_0(\mathfrak{A}(\Lambda^c))' = \pi_0(\mathfrak{A}(\Lambda))''$ by Haag duality. $\square$

Application II: localised repns

Lemma

Suppose that π satisfies the superselection criterion. Then for any cone Λ, there is an equivalent representation $\rho_\Lambda : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ such that $\rho_\Lambda(a) = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda^c)$. Moreover, if $a \in \mathfrak{A}(\Lambda)$, then $\rho_\Lambda(a) \in \pi_0(\mathfrak{A}(\Lambda))''$.

Proof.

By the superselection criterion, there is a unitary $v : \mathcal{H} \to \mathcal{H}_0$ such that $v\pi(a)v^* = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda^c)$. Define $\rho_\Lambda(a) = v\pi_0(a)v^*$. Then $\rho_\Lambda : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ is a representation. Moreover, if $a \in \mathfrak{A}(\Lambda)$ and $b \in \mathfrak{A}(\Lambda^c)$, we have

$$\pi_0(b)\rho_\Lambda(a) = v\pi(b)v^*v\pi(a)v^* = v\pi(ba)v^* = v\pi(ab)v^* = \rho_\Lambda(a)\pi_0(b).$$

The claim then follows by Haag duality. □

Localised representations

By construction, for the “anyon automorphisms” we defined, $\pi_0 \circ \rho(\mathfrak{A}(\Lambda)) \subset \pi_0(\mathfrak{A}(\Lambda))$, and in fact $\rho : \mathfrak{A} \to \mathfrak{A}$.

For an arbitrary representation π satisfying the superselection criterion we can get a unitary equivalent representation $\rho : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ such that $\rho(a) = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda^c)$, where Λ is the localisation region of ρ. But in general $\rho(\mathfrak{A}) \subset \pi_0(\mathfrak{A})$ is not true, i.e. we cannot restrict to endomorphisms of $\mathfrak{A}$.

However, we still get good control over the localisation, namely $\rho(\mathfrak{A}(\Lambda)) \subset \pi_0(\mathfrak{A}(\Lambda))''$.