# A norm bound for 1D local matrices

Let $(\mathcal{M}_j : j \in \mathbb{Z})$ be a family of finite-dimensional Hilbert spaces such that $\mathcal{M}_j = 0$ for sufficiently large $j$ (positive and negative). Thus, the total space $\mathcal{M} := \bigoplus_j \mathcal{M}_j$ is also finite-dimensional. An arbitrary operator $X$ on $\mathcal{M}$ is described by its matrix elements $X_{jk} : \mathcal{M}_k \to \mathcal{M}_j$. We call $X$ *local* (more exactly, 1-*local*) if

$$X_{jk} = 0 \quad \text{if } |j-k| > 1. \tag{1}$$

Let $\Pi_j$ be the projector onto the subspace $\mathcal{M}_j \subseteq \mathcal{M}$ and let

$$\mathcal{M}_{[j,k]} = \bigoplus_{s=j}^{k} \mathcal{M}_s, \qquad \Pi_{[j,k]} = \sum_{s=j}^{k} \Pi_s. \tag{2}$$

**Lemma 1.** *Suppose that, for a given $n$, the following condition holds:*

$$\|\Pi_{[j,j+n]} X \Pi_{[j,j+n]}\| \leqslant 1 \quad \textit{for all } j. \tag{3}$$

*Then $\|X\| \leqslant 1 + O(n^{-2})$.*

**Proof.** Let us choose some real numbers $c_k : k \in \mathbb{Z}$ such that

a) $c_k = 0$ unless $k \in \{0, \ldots, n\}$;

b) $\sum_k c_k^2 = 1$;

c) $\sum_k (c_k - c_{k+1})^2 = O(n^{-2})$.

For example, $c_k = \sqrt{\frac{2}{n+2}} \sin \frac{\pi(k+1)}{n+2}$ if $0 \leqslant k \leqslant n$ and $c_k = 0$ otherwise. Define the operators

$$V_j = \sum_k c_k \Pi_{j+k}, \qquad Y = \sum_j V_j X V_j. \tag{4}$$

Condition (b) on the coefficients $c_k$ implies that $\sum_j V_j^2 = 1$. Furthermore,

$$\begin{aligned} X - Y &= \sum_s \Big(1 - \sum_k c_k c_{k+s}\Big) \Big(\sum_j \Pi_j X \Pi_{j+s}\Big) \\ &= \frac{1}{2} \Big(\sum_k (c_k - c_{k+1})^2\Big) \Big(\sum_j \Pi_j X \Pi_{j+1} + \sum_j \Pi_{j+1} X \Pi_j\Big). \end{aligned} \tag{5}$$

It is clear that $\left\|\sum_j \Pi_j X \Pi_{j+1}\right\| \leqslant \max_j \|X_{j,j+1}\| \leqslant \max_j \|\Pi_{[j,j+1]} X \Pi_{[j,j+1]}\| \leqslant 1$. Thus,

$$\|X - Y\| \leqslant O(n^{-2}). \tag{6}$$

We now show that $\|Y\| \leqslant 1$. For this purpose, the hypothesis of the lemma can be reformulated as follows:

$$\big|\langle \eta | X | \xi \rangle\big| \leqslant \big\||\eta\rangle\big\| \cdot \big\||\xi\rangle\big\| \quad \text{for all } j \text{ and all } |\eta\rangle, |\xi\rangle \in \mathcal{M}_{[j,j+n]}. \tag{7}$$

If $|\eta\rangle, |\xi\rangle \in \mathcal{M}$ are arbitrary unit vectors, then

$$\begin{aligned}
\bigl|\langle\eta|Y|\xi\rangle\bigr| &\leqslant \sum_j \bigl|\langle\eta|V_j X V_j|\xi\rangle\bigr| && (8)\\
&\leqslant \sum_j \bigl\|V_j|\eta\rangle\bigr\| \cdot \bigl\|V_j|\xi\rangle\bigr\| \qquad \text{(due to (7))} && (9)\\
&\leqslant \sqrt{\textstyle\sum_j \bigl\|V_j|\eta\rangle\bigr\|^2}\,\sqrt{\textstyle\sum_j \bigl\|V_j|\xi\rangle\bigr\|^2} \qquad \text{(by Cauchy-Schwarz)} && (10)\\
&= \sqrt{\textstyle\sum_j \langle\eta|V_j^2|\eta\rangle}\,\sqrt{\textstyle\sum_j \langle\xi|V_j^2|\xi\rangle} = 1. && (11)
\end{aligned}$$

The inequalities $\|X-Y\| \leqslant O(n^{-2})$ and $\|Y\| \leqslant 1$ together imply that $\|X\| \leqslant 1 + O(n^{-2})$. □