Problem assignment 1.

Meromorphic Continuation of Eisenstein Series.

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I would like to discuss the notion of **Frechet** representation, as discussed in lectures (Casselman uses the term "Frechet representations of moderate growth").

Bellow I will consider the group $G = SL(2, \mathbf{R})$. For $g \in G$ I define $||g|| = max(||g||_M, ||g^{-1}||_M)$, where $||g||_M$ is just the usual norm on the space of matrices.

1. Let $C(G)$ be the space of continuous functions on G. Show that this is a Frechet space in a natural topology, that the natural representation $\Pi, G, C(G)$ is continuous, but it is not a Frechet representation.

2. Prove the following

Lemma. Let A be a $*$-algebra and $\pi : A \to Op(H)$ its $*$-representation in a Hilbert space H. Suppose we know that the subalgebra $B \subset Op(H)$ contains many compact operators (which means that they do not have common kernel in H).

Show that the representation Π is isomorphic to a sum of irreducible representations $\Pi = \oplus \pi_\kappa$, and each irreducible representation appears in this decomposition with finite multiplicity.

3. Let M be a $(\mathfrak{g}, K)$-module. Suppose we know that for any K-type σ the space M^σ is finite dimensional.

Show that the following conditions are equivalent

(i) M is finitely generated

(ii) M is $\mathcal{Z}(G)$ finite

(iii) M has finite length.

(here $\mathcal{Z}(G)$ is the center of the Universal enveloping algebra $U(\mathfrak{g})$)

Such modules aree called **Harish Chandra modules.**

4. Using Casselman-Wallach theorem prove the following

Statemant. Let (π, G, V) be an asf representation (i.e. an admissible smooth Frechet representation). Then for any Frechet representation ((R, G, E) we have

$$Hom_G(V, E) = Hom_G(V, E^\infty) = Hom_{\mathcal{H}^f}(V^f, E^{\infty f}) == Hom_{(\mathfrak{g},K)}(V^f, E^{\infty f})$$

5. Prove Frobenius reciprocity.

Statement. Let (π, G, V) be an asf representation, $X = \Gamma \backslash G$ an automorphic space. Show that

$$Hom_G(V, F(X)) = Hom_G(V, C^\infty(X)) = Hom_\Gamma(V, \mathbf{C})$$

6. Let (π, G, V) be an asf representation. Define the notion of a contragradient asf representation $\tilde{V} \subset V^*$ and of hermitian dual asf representation $V^+ = \bar{\tilde{V}}$.

Show that for any Banach representation (R, G, E) and a G-moprphism $\nu : V \to E$ the adjoint morphism ν^* maps $(E^*)^\infty$ into $\tilde{V}$.

Show that morphisms $\alpha : V \to C^\infty(X)$ correspond to morphisms $\beta : C_c^\infty(X) \to V^+$.