**Problem assignment 2.**

**Meromorphic Continuation of Eisenstein Series.**

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**1.** Let $(\pi, G, V)$ be a Frechet representation.

Show that the subspace $V^\infty$ of smooth vectors is dense in $V$.

Show that if $\pi$ is admissible then the space $V^f$ of $K$-finite vectors lies in $V^\infty$ and is a $(\mathfrak{g}, K)$-module.

**2.** (i) Show that the function $e^{-x^2} \cdot \exp(i \exp(x^8))$ belongs to $L^2(\mathbf{R})$ but does not lie in the Schwartz space on $\mathbf{R}$.

(ii) Consider on the upper half plane $H$ the function $f = \exp(ix)$.

Show that it lies in $L^2(\mathfrak{S}_T)$ but is not a smooth vector in this space.

(iii) Consider on the upper half plane the function $\phi = y^2 \exp(ix)$ and define the function $f$ on the automorphic space $X = \Gamma \backslash G$ via $f = \sum_{\gamma \in \Gamma_U \backslash \Gamma} \gamma(\phi)$.

Show that this series converges and gives a function $f$ in the space $L^{-2}(X)$ (Hint. It is bounded by a convergent Eisenstein series).

Show that this function $f$ is not a smooth vector in $F^{mod}(X)$.

**3.** Work out the proof of geometric lemma for $SL(2, \mathbf{Z})$, namely that $C \cdot E = 1 + D$, where $D$ skew commutes with the action of $M$, i.e. $\rho(m)D = D\rho(m^{-1})$.

**4.** Prove basic properties of holomorphic families of morphisms.

(i) Suppose $\nu(s) : F \to W$ and $\lambda(s) : W \to V$ are holomorphic families. Then the family $\mu(s) = \lambda(s) \cdot \nu(s) : F \to V$ is also a holomorphic family of morphisms.

(ii) Let $\nu(s) : F \to W$ be a holomorphic family of morphisms of Banach spaces. Show that it is continuous and moreover for every point $a \in S$ we can expand near $a$ $\nu(s) = \sum_\alpha B_\alpha (s-a)^\alpha$, where $||B_\alpha|| \leq C r^{|\alpha|}$ for some $r$.

**5.** Let $\Xi_s$ be a holomorphic system of linear equations in a finite-dimensional vector space $L$. For every $s \in S$ consider the number $k_s = \dim(Sol(\Xi_s))$ (it can be $-\infty, 0, 1, ...$).

Show that the function $s \mapsto k_s$ is constant almost everywhere and equals some number $k$.

Show that if $k \geq 0$ we can add to the system $\Xi$ $k$ additional equations, independent of $s$, such that the resulting system of equations $\Xi'$ almost everywhere has unique solution $v(s)$ and this solution is a meromorphic function in $s$.

**6.** Let $\nu(s) : F \to W$ be a holomorphic family of morphisms of Hilbertian spaces. Suppose we know that at a point $a \in S$ the operator $\nu(a)$ has left inverse modulo compact operators (this means that there exists an operator $I : W \to F$ such that $I \cdot \nu(a) = 1 + C$, where $C$ is a compact operator).

Show that the system of equations $\Xi_s : \nu(s)(v) = 0$ is of finite type near the point $a$.

**7.** Let $G = SL(2, \mathbf{R}), \Gamma = SL(2, \mathbf{Z}),\ X = \Gamma \backslash G$ the automorphic space.

Construct a weight function $w$ on $X$ such that on a Siegel domain $\mathfrak{S}_T$ it will be comparable to function $y$ (imaginary coordinate on the upper half plane which we consider as a function on $U \backslash G$ and hence a function on $Z_B$).

We consider the basic spaces $L_k = L^2(X, w^{2k}\mu)$.

By definition $F^{mod}(X) = \bigcup L_k$ is the space of functions of moderate growth on $X$ and $F^{rd}(X) = \bigcap L_k$ is the space of rapidly decreasing functions on $X$.

Show that if $T$ is small, then the natural morphism $i : L_k(X) \to L_k(\mathfrak{S}_T)$ is a closed imbedding with controllable norms (this means that $c||v|| \leq ||i(v)|| \leq C||v||$ for all vectors $v \in L_k(X)$).