1. Let \((\pi, G, V)\) be a Frechet representation.
   Show that the subspace \(V^\infty\) of smooth vectors is dense in \(V\).
   Show that if \(\pi\) is admissible then the space \(V^f\) of \(K\)-finite vectors lies in \(V^\infty\) and is a \((g, K)\)-module.

2. (i) Show that the function \(e^{-x^2} \cdot \exp(i \exp(x^8))\) belongs to \(L^2(\mathbb{R})\) but does not lie in the Schwartz space on \(\mathbb{R}\).
   (ii) Consider on the upper half plane \(H\) the function \(f = \exp(ix)\).
   Show that it lies in \(L^2(S)\) but is not a smooth vector in this space.
   (iii) Consider on the upper half plane the function \(\phi = y^2 \exp(ix)\) and define the function \(f\) on the automorphic space \(X = \Gamma \backslash G\) via \(f = \sum_{\gamma \in \Gamma \backslash \Gamma} \gamma(\phi)\).
   Show that this series converges and gives a function \(f\) in the space \(L^{-2}(X)\) (Hint. It is bounded by a convergent Eisenstein series).
   Show that this function \(f\) is not a smooth vector in \(F^{mod}(X)\).

3. Work out the proof of geometric lemma for \(SL(2, \mathbb{Z})\), namely that \(C \cdot E = 1 + D\), where \(D\) skew commutes with the action of \(M\), i.e. \(\rho(m)D = D\rho(m^{-1})\).

4. Prove basic properties of holomorphic families of morphisms.
   (i) Suppose \(\nu(s) : F \to W\) and \(\lambda(s) : W \to V\) are holomorphic families.
   Then the family \(\mu(s) = \lambda(s) \cdot \nu(s) : F \to V\) is also a holomorphic family of morphisms.
   (ii) Let \(\nu(s) : F \to W\) be a holomorphic family of morphisms of Banach spaces. Show that it is continuous and moreover for every point \(a \in S\) we can expand near \(a\) \(\nu(s) = \sum B_a(s-a)^{\alpha}\), where \(\|B_a\| \leq C r^{\alpha}\) for some \(r\).

5. Let \(\Xi_s\) be a holomorphic system of linear equations in a finite-dimensional vector space \(L\). For every \(s \in S\) consider the number \(k_s = \dim(Sol(\Xi_s))\) (it can be \(-\infty, 0, 1, ...\)).
   Show that the function \(s \mapsto k_s\) is constant almost everywhere and equals some number \(k\).
   Show that if \(k \geq 0\) we can add to the system \(\Xi\) \(k\) additional equations, independent of \(s\), such that the resulting system of equations \(\Xi’\) almost everywhere has unique solution \(v(s)\) and this solution is a meromorphic function in \(s\).

6. Let \(\nu(s) : F \to W\) be a holomorphic family of morphisms of Hilbertian spaces. Suppose we know that at a point \(a \in S\) the operator \(\nu(a)\) has left inverse modulo compact operators (this means that there exists an operator \(I : W \to F\) such that \(I \cdot \nu(a) = 1 + C\), where \(C\) is a compact operator).
   Show that the system of equations \(\Xi_a : \nu(s)(v) = 0\) is of finite type near the point \(a\).
7. Let $G = SL(2, \mathbb{R}), \Gamma = SL(2, \mathbb{Z}), X = \Gamma \backslash G$ the automorphic space.

Construct a weight function $w$ on $X$ such that on a Siegel domain $\mathcal{S}_T$ it will be comparable to function $y$ (imaginary coordinate on the upper half plane which we consider as a function on $U \backslash G$ and hence a function on $\mathcal{Z}_B$).

We consider the basic spaces $L_k = L^2(X, w^k \mu)$.

By definition $F^{mod}(X) = \bigcup L_k$ is the space of functions of moderate growth on $X$ and $F^{rd}(X) = \bigcap L_k$ is the space of rapidly decreasing functions on $X$.

Show that if $T$ is small, then the natural morphism $i : L_k(X) \to L_k(\mathcal{S}_T)$ is a closed imbedding with controllable norms (this means that $c||v|| \leq ||i(v)|| \leq C||v||$ for all vectors $v \in L_k(X)$).