CATEGORICAL $\mathfrak{sl}_2$ ACTIONS

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1. Introduction

1.1. **Actions of $\mathfrak{sl}_2$ on categories.** A action of $\mathfrak{sl}_2$ on a finite-dimensional $\mathbb{C}$-vector space V consists of a direct sum decomposition $V = \oplus V(\lambda)$ into weight spaces, together with linear maps

$$e(\lambda) : V(\lambda-1) \to V(\lambda+1) \text{ and } f(\lambda) : V(\lambda+1) \to V(\lambda-1)$$

satisfying the condition

$$e(\lambda-1)f(\lambda-1) - f(\lambda+1)e(\lambda+1) = \lambda \mathrm{I}_{V(\lambda)}. \tag{1}$$

Such an action automatically integrates to the group SL_2. In particular, the reflection element

$$t = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in SL_2$$

acts on V, inducing an isomorphism $V(-\lambda) \to V(\lambda)$.

A first pass at a categorification of this structure involves replacing vector spaces with categories and linear maps with functors. Thus, a naïve categorification of a finite dimensional $\mathfrak{sl}_2$ module consists of a sequence of categories $\mathcal{D}(\lambda)$, together with functors

$$\mathsf{E}(\lambda) : \mathcal{D}(\lambda-1) \to \mathcal{D}(\lambda+1) \text{ and } \mathsf{F}(\lambda) : \mathcal{D}(\lambda+1) \to \mathcal{D}(\lambda-1)$$

between them. These functors should satisfy a categorical version of (1) above,

$$\mathsf{E}(\lambda-1) \circ \mathsf{F}(\lambda-1) \cong \mathrm{I}_{\mathcal{D}(\lambda)}^{\oplus\lambda} \oplus \mathsf{F}(\lambda+1) \circ \mathsf{E}(\lambda+1), \quad \text{for } \lambda \geq 0, \tag{2}$$

and an analogous condition when $\lambda \leq 0$. The sense in which this is naïve is that ideally there should be specified natural transformations which induce the isomorphisms (2).

2. Chuang-Rouquier's definition of $\mathfrak{sl}_2$-categorification

In order to get a good theory of $\mathfrak{sl}_2$-categorification, we need to define the algebraic structure arising from natural transformations between various compositions of the functors E and F. The first such definition, due to Joe Chuang and Raphael Rouquier [CR], is given below. (In the definition, as well as in some later parts of the abstract, we will omit the λ from the notation, writing E and F instead of $\mathsf{E}(\lambda)$ and $\mathsf{F}(\lambda)$.

Definition 2.1. *An $\mathfrak{sl}_2$ categorification consists of a finite length abelian category $\mathcal{A}$, together with exact functors $\mathsf{E}, \mathsf{F} : \mathcal{A} \to \mathcal{A}$ such that:*

- (i) E *is a left and right adjoint to* F*;*
- (ii) *The action of* $[\mathsf{E}]$ *and* $[\mathsf{F}]$ *on* $V = K_{\mathbb{C}}(\mathcal{A})$ *induces a locally finite action of* $\mathfrak{sl}_2$*;*
- (iii) *We have a decomposition* $\mathcal{A} = \bigoplus_{\lambda\in\mathbb{Z}} \mathcal{A}_\lambda$ *such that* $K_{\mathbb{C}}(\mathcal{A}_\lambda) = V_\lambda$ *is a weight space of* V*.*

We also require natural transformations $X : \mathsf{E} \to \mathsf{E}$ *and* $T : \mathsf{EE} \to \mathsf{EE}$ *such that*

- (i) $T^2 = \mathrm{I}_{\mathsf{EE}}$*;*
- (ii) $(T\mathrm{I}_{\mathsf{E}}) \circ (\mathrm{I}_{\mathsf{E}}T) \circ (T\mathrm{I}_{\mathsf{E}}) = (\mathrm{I}_{\mathsf{E}}T) \circ (T\mathrm{I}_{\mathsf{E}}) \circ (\mathrm{I}_{\mathsf{E}}T)$ *in* $\mathrm{End}(\mathsf{E}^3)$*;*
- (iii) $T \circ (\mathrm{I}_E X) = (X\mathrm{I}_E) \circ T - \mathrm{I}_{EE}$*;*
- (iv) $X_M \in \mathrm{End}(\mathsf{E}M)$ *is nilpotent for all objects* $M \in \mathcal{A}$*.*

It follows that endomorphisms X and T induce an action of the *degenerate affine Hecke algebra* of GL_n on E^n (and, by adjunction, on F^n.) As a consequence of the definition, Chuang-Rouquier prove that the functor E^n is isomorphic to the direct sum of $n!$ copies of a single functor $\mathsf{E}^{(n)}$. Similarly, by adjunction, the functor F^n is isomorphic to $n!$ copies of a single functor $\mathsf{F}^{(n)}$. Thus $\mathsf{E}^{(n)}$ and $\mathsf{F}^{(n)}$ naturally categorify the divided powers $e^{(n)} = \frac{e^n}{n!}$ and $f^{(n)} = \frac{f^n}{n!}$. Chuang-Rouquier then define a complex $\Theta(\lambda)$ of functors, which they call the *Rickard complex.* The terms of the Rickard complex are

$$\Theta(\lambda)_d = \mathsf{E}^{(\lambda+d)}\mathsf{F}^{(d)},$$

and the differential $\delta : \Theta(\lambda)_d \to \Theta(\lambda)_{d-1}$ is built from the adjunction morphism $\mathsf{EF} \to \mathrm{I}$, see [CR].

Theorem 2.2. *(Chuang-Rouquier) The functor $\Theta(\lambda)$ defines an equivalence of categories*

$$\Theta(\lambda) : D^b(\mathcal{A}_{-\lambda}) \simeq D^b(\mathcal{A}_\lambda).$$

Futhermore, Chuang and Rouquier construct an explicit $\mathfrak{sl}_2$ categorification using direct summands of induction and restriction functors between symmetric groups. As a corollary of the above theorem, they are then able to prove Broue's abelian defect conjecture for symmetric groups.

3. Geometric examples of $\mathfrak{sl}_2$ categorification

There are geometric examples of categorical $\mathfrak{sl}_2$ actions which do not quite satisfy the hypotheses in the Chuang-Rouquier definition above, essentially because the underlying weight space categories are not abelian (though they are triangulated) and the degenerate affine Hecke algebra does not act naturally on E^n (though the *nil affine Hecke algebra* does.) In these cases, the Chuang-Rouquier definition must be modified slightly.

3.1. **Categorical $\mathfrak{sl}_2$ actions.** We begin by giving a modified definition of $\mathfrak{sl}_2$ categorification which was introduced in joint work with Sabin Cautis and Joel Kamnitzer [CKL1],[CKL2],[CKL3]. Then we will discuss the basic geometric example, which involves cotangent bundles to Grassmanians. Let $\Bbbk$ be a field. We denote by $\mathbb{P}^r$ the projective space of lines in an r-dimensional $\mathbb{C}$ vector space, by $\mathbb{G}(r_1, r_1 + r_2))$ the Grassmanian of r_1-planes in $r_1 + r_2$ space, and by$H^*(\mathbb{G}(r_1, r_1 + r_2))$ the singular cohomology of the Grassmanian, with it's grading shifted to be symmetric about 0.

A **categorical $\mathfrak{sl}_2$ action** consists of the following data:

- A sequence of $\Bbbk$-linear, $\mathbb{Z}$-graded, additive categories $\mathcal{D}(-N), \dots, \mathcal{D}(N)$ which are idempotent complete. "Graded" means that each category $\mathcal{D}(\lambda)$ has a shift functor $\langle \cdot \rangle$ which is an equivalence.
- Functors
$$\mathsf{E}^{(r)}(\lambda) : \mathcal{D}(\lambda - r) \to \mathcal{D}(\lambda + r) \text{ and } \mathsf{F}^{(r)}(\lambda) : \mathcal{D}(\lambda + r) \to \mathcal{D}(\lambda - r)$$
for $r \geq 0$ and $\lambda \in \mathbb{Z}$.
- Morphisms
$$\eta : \mathrm{I} \to \mathsf{F}^{(r)}(\lambda)\mathsf{E}^{(r)}(\lambda)\langle r\lambda \rangle \text{ and } \eta : \mathrm{I} \to \mathsf{E}^{(r)}(\lambda)\mathsf{F}^{(r)}(\lambda)\langle -r\lambda \rangle$$
$$\varepsilon : \mathsf{F}^{(r)}(\lambda)\mathsf{E}^{(r)}(\lambda) \to \mathrm{I}\langle r\lambda \rangle \text{ and } \varepsilon : \mathsf{E}^{(r)}(\lambda)\mathsf{F}^{(r)}(\lambda) \to \mathrm{I}\langle -r\lambda \rangle.$$
- Morphisms
$$\iota : \mathsf{E}^{(r+1)}(\lambda)\langle r \rangle \to \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \text{ and } \pi : \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \to \mathsf{E}^{(r+1)}(\lambda)\langle -r \rangle.$$
- Morphisms
$$X(\lambda) : \mathsf{E}(\lambda)\langle -1 \rangle \to \mathsf{E}(\lambda)\langle 1 \rangle \text{ and } T(\lambda) : \mathsf{E}(\lambda+1)\mathsf{E}(\lambda-1)\langle 1 \rangle \to \mathsf{E}(\lambda+1)\mathsf{E}(\lambda-1)\langle -1 \rangle.$$

On this data we impose the following additional conditions:

- The morphisms η and ε are units and co-units of adjunctions

(i) $\mathsf{E}^{(r)}(\lambda)_R = \mathsf{F}^{(r)}(\lambda)\langle r\lambda\rangle$ for $r \geq 0$
(ii) $\mathsf{E}^{(r)}(\lambda)_L = \mathsf{F}^{(r)}(\lambda)\langle -r\lambda\rangle$ for $r \geq 0$

- E's compose as
$$\mathsf{E}^{(r_2)}(\lambda + r_1)\mathsf{E}^{(r_1)}(\lambda - r_2) \cong \mathsf{E}^{(r_1+r_2)}(\lambda) \otimes_{\Bbbk} H^*(\mathbb{G}(r_1, r_1 + r_2))$$
For example,
$$\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1) \cong \mathsf{E}^{(2)}(\lambda)\langle -1\rangle \oplus \mathsf{E}^{(2)}(\lambda)\langle 1\rangle.$$
(By adjointness the F's compose similarly.) In the case $r_1 = r$ and $r_2 = 1$ we also require that the maps
$$\oplus_{i=0}^{r}(X(\lambda + r)^i I) \circ \iota\langle -2i\rangle : \mathsf{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r) \to \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1)$$
and
$$\oplus_{i=0}^{r}\pi\langle 2i\rangle \circ (X(\lambda + r)^i I) : \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \to \mathsf{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r)$$
are isomorphisms. We also have the analogous condition when $r_1 = 1$ and $r_2 = r$.
- If $\lambda \leq 0$ then
$$\mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1) \cong \mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}).$$
The isomorphism is induced by
$$\sigma + \sum_{j=0}^{-\lambda-1}(IX(\lambda + 1)^j) \circ \eta : \mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}) \xrightarrow{\sim} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)$$
where σ is the composition of maps
$$\begin{aligned}\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) &\xrightarrow{\eta II} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1)\langle \lambda + 1\rangle \\ &\xrightarrow{IT(\lambda)\mathrm{I}} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1)\langle \lambda - 1\rangle \\ &\xrightarrow{II\epsilon} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1).\end{aligned}$$
Similarly, if $\lambda \geq 0$, then
$$\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \cong \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{\lambda-1}),$$
with the isomorphism induced as above.
- The X's and T's satisfy the nil affine Hecke relations:
(i) $T(\lambda)^2 = 0$
(ii) $(IT(\lambda - 1)) \circ (T(\lambda + 1)I) \circ (IT(\lambda - 1)) = (T(\lambda + 1)I) \circ (IT(\lambda - 1)) \circ (T(\lambda + 1)I)$ as endomorphisms of $\mathsf{E}(\lambda - 2)\mathsf{E}(\lambda)\mathsf{E}(\lambda + 2)$.
(iii) $(X(\lambda + 1)I) \circ T(\lambda) - T(\lambda) \circ (IX(\lambda - 1)) = I = -(IX(\lambda - 1)) \circ T(\lambda) + T(\lambda) \circ (X(\lambda + 1))$ as endomorphisms of $\mathsf{E}(\lambda - 1)\mathsf{E}(\lambda + 1)$.
- For $r \geq 0$, we have $\mathrm{Hom}(\mathsf{E}^{(r)}(\lambda), \mathsf{E}^{(r)}(\lambda)\langle i\rangle) = 0$ if $i < 0$ and $\mathrm{End}(\mathsf{E}^{(r)}(\lambda)) = \Bbbk \cdot \mathrm{I}$.

Given a categorical $\mathfrak{sl}_2$ action, for each $\lambda \geq 0$ we may construct the Rickard complex [CKL2]
$$\Theta_* : \mathcal{D}(\lambda) \to \mathcal{D}(-\lambda).$$
The terms in the complex are
$$\Theta_s = \mathsf{F}^{(\lambda+s)}(s)\mathsf{E}^{(s)}(\lambda + s)\langle -s\rangle,$$
where $s = 0, \ldots, (N - \lambda)/2$. The differential $d_s : \Theta_s \to \Theta_{s-1}$ is given by the composition of maps
$$\mathsf{F}^{(\lambda+s)}\mathsf{E}^{(s)}\langle -s\rangle \xrightarrow{\iota\iota} \mathsf{F}^{(\lambda+s-1)}\mathsf{F}\mathsf{E}\mathsf{E}^{(s-1)}\langle -(\lambda + s - 1) - (s - 1) - s\rangle \xrightarrow{\varepsilon} \mathsf{F}^{(\lambda+s-1)}\mathsf{E}^{(s-1)}\langle -s + 1\rangle.$$
Then we have the following theorem, proved in [CKL2].

Theorem 3.1. *Suppose the underlying weight space categories $\mathcal{D}(\lambda)$ are triangulated. Then complex Θ_* has a unique convolution T, and $\mathsf{T} : \mathcal{D}(-\lambda) \longrightarrow \mathcal{D}(\lambda)$ is an equivalence of triangulated categories.*

3.2. **A Geometric Example.** The basic example of a categorical $\mathfrak{sl}_2$ action comes from Grassmanian geometry, and we refer to [CKL2] for complete details.

Fix $N > 0$. For our weight spaces we will take the derived category of coherent sheaves on the cotangent bundle to the Grassmannian $T^\star\mathbb{G}(k,N)$. We use shorthand $Y(\lambda) = T^\star\mathbb{G}(k,N)$, where $k = (N-\lambda)/2$. These spaces have a particularly nice geometric description,

$$T^\star\mathbb{G}(k,N) \cong \{(X,V) : X \in \mathrm{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } \mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0\},$$

where $\mathrm{End}(\mathbb{C}^N)$ denotes the space of complex $N \times N$ matrices. (The notation $\mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0$ means that $X(\mathbb{C}^n) \subset V$ and that $X(V) = 0$.) Forgetting X corresponds to the projection $T^\star\mathbb{G}(k,N) \to \mathbb{G}(k,N)$ while forgetting V gives a resolution of the variety

$$\{X \in \mathrm{End}(\mathbb{C}^N) : X^2 = 0 \text{ and } \mathrm{rank}(X) \le \min(k, N-k)\}.$$

On $T^\star\mathbb{G}(k,N)$ we have the tautological rank k vector bundle V as well as the quotient $\mathbb{C}^N/V$.

To describe the kernels $\mathcal{E}$ and $\mathcal{F}$ we will need the correspondences

$$W^r(\lambda) \subset T^\star\mathbb{G}(k+r/2,N) \times T^\star\mathbb{G}(k-r/2,N)$$

defined by

$$\begin{aligned} W^r(\lambda) := \{(X,V,V') :& X \in \mathrm{End}(\mathbb{C}^N), \dim(V) = k + \frac{r}{2}, \dim(V') = k - \frac{r}{2}, \\ & 0 \subset V' \subset V \subset \mathbb{C}^N,\ \ \mathbb{C}^N \xrightarrow{X} V' \text{ ,and } V \xrightarrow{X} 0\}. \end{aligned}$$

(Here, as before, λ and k are related by the equation $k = (N-\lambda)/2$).

There are two natural projections $\pi_1 : (X,V,V') \mapsto (X,V)$ and $\pi_2 : (X,V,V') \mapsto (X,V')$ from $W^r(\lambda)$ to $Y(\lambda-r)$ and $Y(\lambda+r)$ respectively. Together they give us an embedding

$$(\pi_1,\pi_2) : W^r(\lambda) \subset Y(\lambda-r) \times Y(\lambda+r).$$

On $W^r(\lambda)$ we have two natural tautological bundles, namely $V := \pi_1^*(V)$ and $V' := \pi_2^*(V)$, where the prime on the V' indicates that the vector bundle is the pullback of the tautological bundle by the second projection. We also have natural inclusions

$$0 \subset V' \subset V \subset \mathbb{C}^N \cong \mathcal{O}_{W^r(\lambda)}^{\oplus N}.$$

We now define the kernel $\mathcal{E}^{(r)}(\lambda) \in D(Y(\lambda-r) \times Y(\lambda+r))$ by

$$\mathcal{E}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(\mathbb{C}^N/V')^{-r}\det(V)^r\{\frac{r(N-\lambda-r)}{2}\}.$$

Similarly, the kernel $\mathcal{F}^{(r)}(\lambda) \in D(Y(\lambda+r) \times Y(\lambda-r))$ is defined by

$$\mathcal{F}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(V'/V)^{\lambda}\{\frac{r(N+\lambda-r)}{2}\}.$$

These kernels define functors (Fourier-Mukai transforms) $\mathsf{E}^{(k)}$ and $\mathsf{F}^{(k)}$, and in [CKL2] we define natural transformations which enhance these functors to a full categorical $\mathfrak{sl}_2$ action.

As a result, we may define the Rickard complex Θ. Convolution with this complex gives new equivalences of triangulated categories between categories corresponding to opposite $\mathfrak{sl}_2$ weight spaces.

Corollary 3.2. [CKL3] *The complex Θ defines an equivalence between derived categories of coherent sheaves of cotangent bundles to dual Grassmanians*

$$\Theta : D(T^*(G(k,N)) \simeq D(T^*(G(N-k,N)).$$

4. Further Develepements

The notion of $\mathfrak{sl}_2$-categorification goes back at least to the paper [BFK], which inspired much of the subsequent work on algebraic aspects of categorification. After the seminal contribution [CR], which contains several algebraic examples of $\mathfrak{sl}_2$ categorifications, various geometric aspects of categorical $\mathfrak{sl}_2$ representation theory were develeoped in [CKL1], [CKL2], [CKL3].

On the other hand, it is quite natural to categorify the entire quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, rather than just the finite dimensional representations. This has been accomplished by Rouquier [R], and by Lauda [L]. Moreover, the entire story can be generalized and repeated, with the lead actor $\mathfrak{sl}_2$ replaced by an arbitrary symmetrizable Kac-Moody Lie algbera $\mathfrak{g}$. This is the subject of the significant work of Khovanov-Lauda [KL] and, independently, Rouquier [R].

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