

CATEGORICAL \mathfrak{sl}_2 ACTIONS

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1. INTRODUCTION

1.1. Actions of \mathfrak{sl}_2 on categories. A action of \mathfrak{sl}_2 on a finite-dimensional \mathbb{C} -vector space V consists of a direct sum decomposition $V = \oplus V(\lambda)$ into weight spaces, together with linear maps

$$e(\lambda) : V(\lambda - 1) \rightarrow V(\lambda + 1) \text{ and } f(\lambda) : V(\lambda + 1) \rightarrow V(\lambda - 1)$$

satisfying the condition

$$(1) \quad e(\lambda - 1)f(\lambda - 1) - f(\lambda + 1)e(\lambda + 1) = \lambda I_{V(\lambda)}.$$

Such an action automatically integrates to the group SL_2 . In particular, the reflection element

$$t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2$$

acts on V , inducing an isomorphism $V(-\lambda) \rightarrow V(\lambda)$.

A first pass at a categorification of this structure involves replacing vector spaces with categories and linear maps with functors. Thus, a naïve categorification of a finite dimensional \mathfrak{sl}_2 module consists of a sequence of categories $\mathcal{D}(\lambda)$, together with functors

$$E(\lambda) : \mathcal{D}(\lambda - 1) \rightarrow \mathcal{D}(\lambda + 1) \text{ and } F(\lambda) : \mathcal{D}(\lambda + 1) \rightarrow \mathcal{D}(\lambda - 1)$$

between them. These functors should satisfy a categorical version of (1) above,

$$(2) \quad E(\lambda - 1) \circ F(\lambda - 1) \cong I_{\mathcal{D}(\lambda)}^{\oplus \lambda} \oplus F(\lambda + 1) \circ E(\lambda + 1), \quad \text{for } \lambda \geq 0,$$

and an analogous condition when $\lambda \leq 0$. The sense in which this is naïve is that ideally there should be specified natural transformations which induce the isomorphisms (2).

2. CHUANG-ROUQUIER'S DEFINITION OF \mathfrak{sl}_2 -CATEGORIFICATION

In order to get a good theory of \mathfrak{sl}_2 -categorification, we need to define the algebraic structure arising from natural transformations between various compositions of the functors E and F . The first such definition, due to Joe Chuang and Raphael Rouquier [CR], is given below. (In the definition, as well as in some later parts of the abstract, we will omit the λ from the notation, writing E and F instead of $E(\lambda)$ and $F(\lambda)$).

Definition 2.1. *An \mathfrak{sl}_2 categorification consists of a finite length abelian category \mathcal{A} , together with exact functors $E, F : \mathcal{A} \rightarrow \mathcal{A}$ such that:*

- (i) *E is a left and right adjoint to F ;*
- (ii) *The action of $[E]$ and $[F]$ on $V = K_{\mathbb{C}}(\mathcal{A})$ induces a locally finite action of \mathfrak{sl}_2 ;*
- (iii) *We have a decomposition $\mathcal{A} = \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{A}_{\lambda}$ such that $K_{\mathbb{C}}(\mathcal{A}_{\lambda}) = V_{\lambda}$ is a weight space of V .*

We also require natural transformations $X : E \rightarrow E$ and $T : EE \rightarrow EE$ such that

- (i) *$T^2 = I_{EE}$;*
- (ii) *$(TI_E) \circ (I_ET) \circ (TI_E) = (I_ET) \circ (TI_E) \circ (I_ET)$ in $\text{End}(E^3)$;*
- (iii) *$T \circ (I_EX) = (XI_E) \circ T - I_{EE}$;*
- (iv) *$X_M \in \text{End}(EM)$ is nilpotent for all objects $M \in \mathcal{A}$.*

It follows that endomorphisms X and T induce an action of the *degenerate affine Hecke algebra* of GL_n on E^n (and, by adjunction, on F^n .) As a consequence of the definition, Chuang-Rouquier prove that the functor E^n is isomorphic to the direct sum of $n!$ copies of a single functor $E^{(n)}$. Similarly, by adjunction, the functor F^n is isomorphic to $n!$ copies of a single functor $F^{(n)}$. Thus $E^{(n)}$ and $F^{(n)}$ naturally categorify the divided powers $e^{(n)} = \frac{e^n}{n!}$ and $f^{(n)} = \frac{f^n}{n!}$. Chuang-Rouquier then define a complex $\Theta(\lambda)$ of functors, which they call the *Rickard complex*. The terms of the Rickard complex are

$$\Theta(\lambda)_d = E^{(\lambda+d)} F^{(d)},$$

and the differential $\delta : \Theta(\lambda)_d \rightarrow \Theta(\lambda)_{d-1}$ is built from the adjunction morphism $EF \rightarrow I$, see [CR].

Theorem 2.2. (*Chuang-Rouquier*) *The functor $\Theta(\lambda)$ defines an equivalence of categories*

$$\Theta(\lambda) : D^b(\mathcal{A}_{-\lambda}) \simeq D^b(\mathcal{A}_\lambda).$$

Futhermore, Chuang and Rouquier construct an explicit \mathfrak{sl}_2 categorification using direct summands of induction and restriction functors between symmetric groups. As a corollary of the above theorem, they are then able to prove Broue's abelian defect conjecture for symmetric groups.

3. GEOMETRIC EXAMPLES OF \mathfrak{sl}_2 CATEGORIFICATION

There are geometric examples of categorical \mathfrak{sl}_2 actions which do not quite satisfy the hypotheses in the Chuang-Rouquier definition above, essentially because the underlying weight space categories are not abelian (though they are triangulated) and the degenerate affine Hecke algebra does not act naturally on E^n (though the *nil affine Hecke algebra* does.) In these cases, the Chuang-Rouquier definition must be modified slightly.

3.1. Categorical \mathfrak{sl}_2 actions. We begin by giving a modified definition of \mathfrak{sl}_2 categorification which was introduced in joint work with Sabin Cautis and Joel Kamnitzer [CKL1],[CKL2],[CKL3]. Then we will discuss the basic geometric example, which involves cotangent bundles to Grassmanians. Let \mathbb{k} be a field. We denote by \mathbb{P}^r the projective space of lines in an r -dimensional \mathbb{C} vector space, by $\mathbb{G}(r_1, r_1 + r_2)$ the Grassmanian of r_1 -planes in $r_1 + r_2$ space, and by $H^*(\mathbb{G}(r_1, r_1 + r_2))$ the singular cohomology of the Grassmanian, with its grading shifted to be symmetric about 0.

A **categorical \mathfrak{sl}_2 action** consists of the following data:

- A sequence of \mathbb{k} -linear, \mathbb{Z} -graded, additive categories $\mathcal{D}(-N), \dots, \mathcal{D}(N)$ which are idempotent complete. “Graded” means that each category $\mathcal{D}(\lambda)$ has a shift functor $\langle \cdot \rangle$ which is an equivalence.
- Functors

$$E^{(r)}(\lambda) : \mathcal{D}(\lambda - r) \rightarrow \mathcal{D}(\lambda + r) \text{ and } F^{(r)}(\lambda) : \mathcal{D}(\lambda + r) \rightarrow \mathcal{D}(\lambda - r)$$

for $r \geq 0$ and $\lambda \in \mathbb{Z}$.

- Morphisms

$$\eta : I \rightarrow F^{(r)}(\lambda) E^{(r)}(\lambda) \langle r\lambda \rangle \text{ and } \eta : I \rightarrow E^{(r)}(\lambda) F^{(r)}(\lambda) \langle -r\lambda \rangle$$

$$\varepsilon : F^{(r)}(\lambda) E^{(r)}(\lambda) \rightarrow I \langle r\lambda \rangle \text{ and } \varepsilon : E^{(r)}(\lambda) F^{(r)}(\lambda) \rightarrow I \langle -r\lambda \rangle.$$

- Morphisms

$$\iota : E^{(r+1)}(\lambda) \langle r \rangle \rightarrow E(\lambda + r) E^{(r)}(\lambda - 1) \text{ and } \pi : E(\lambda + r) E^{(r)}(\lambda - 1) \rightarrow E^{(r+1)}(\lambda) \langle -r \rangle.$$

- Morphisms

$$X(\lambda) : E(\lambda) \langle -1 \rangle \rightarrow E(\lambda) \langle 1 \rangle \text{ and } T(\lambda) : E(\lambda + 1) E(\lambda - 1) \langle 1 \rangle \rightarrow E(\lambda + 1) E(\lambda - 1) \langle -1 \rangle.$$

On this data we impose the following additional conditions:

- The morphisms η and ε are units and co-units of adjunctions

- (i) $E^{(r)}(\lambda)_R = F^{(r)}(\lambda)\langle r\lambda \rangle$ for $r \geq 0$
- (ii) $E^{(r)}(\lambda)_L = F^{(r)}(\lambda)\langle -r\lambda \rangle$ for $r \geq 0$
- E 's compose as

$$E^{(r_2)}(\lambda + r_1)E^{(r_1)}(\lambda - r_2) \cong E^{(r_1+r_2)}(\lambda) \otimes_{\mathbb{k}} H^*(\mathbb{G}(r_1, r_1 + r_2))$$

For example,

$$E(\lambda + 1)E(\lambda - 1) \cong E^{(2)}(\lambda)\langle -1 \rangle \oplus E^{(2)}(\lambda)\langle 1 \rangle.$$

(By adjointness the F 's compose similarly.) In the case $r_1 = r$ and $r_2 = 1$ we also require that the maps

$$\oplus_{i=0}^r (X(\lambda + r)^i I) \circ \iota \langle -2i \rangle : E^{(r+1)}(\lambda) \otimes_{\mathbb{k}} H^*(\mathbb{P}^r) \rightarrow E(\lambda + r)E^{(r)}(\lambda - 1)$$

and

$$\oplus_{i=0}^r \pi \langle 2i \rangle \circ (X(\lambda + r)^i I) : E(\lambda + r)E^{(r)}(\lambda - 1) \rightarrow E^{(r+1)}(\lambda) \otimes_{\mathbb{k}} H^*(\mathbb{P}^r)$$

are isomorphisms. We also have the analogous condition when $r_1 = 1$ and $r_2 = r$.

- If $\lambda \leq 0$ then

$$F(\lambda + 1)E(\lambda + 1) \cong E(\lambda - 1)F(\lambda - 1) \oplus I \otimes_{\mathbb{k}} H^*(\mathbb{P}^{-\lambda-1}).$$

The isomorphism is induced by

$$\sigma + \sum_{j=0}^{-\lambda-1} (IX(\lambda + 1)^j) \circ \eta : E(\lambda - 1)F(\lambda - 1) \oplus I \otimes_{\mathbb{k}} H^*(\mathbb{P}^{-\lambda-1}) \xrightarrow{\sim} F(\lambda + 1)E(\lambda + 1)$$

where σ is the composition of maps

$$\begin{aligned} E(\lambda - 1)F(\lambda - 1) &\xrightarrow{\eta II} F(\lambda + 1)E(\lambda + 1)E(\lambda - 1)F(\lambda - 1)\langle \lambda + 1 \rangle \\ &\xrightarrow{IT(\lambda)I} F(\lambda + 1)E(\lambda + 1)E(\lambda - 1)F(\lambda - 1)\langle \lambda - 1 \rangle \\ &\xrightarrow{II\epsilon} F(\lambda + 1)E(\lambda + 1). \end{aligned}$$

Similarly, if $\lambda \geq 0$, then

$$E(\lambda - 1)F(\lambda - 1) \cong F(\lambda + 1)E(\lambda + 1) \oplus I \otimes_{\mathbb{k}} H^*(\mathbb{P}^{\lambda-1}),$$

with the isomorphism induced as above.

- The X 's and T 's satisfy the nil affine Hecke relations:
 - (i) $T(\lambda)^2 = 0$
 - (ii) $(IT(\lambda - 1)) \circ (T(\lambda + 1)I) \circ (IT(\lambda - 1)) = (T(\lambda + 1)I) \circ (IT(\lambda - 1)) \circ (T(\lambda + 1)I)$ as endomorphisms of $E(\lambda - 2)E(\lambda)E(\lambda + 2)$.
 - (iii) $(X(\lambda + 1)I) \circ T(\lambda) - T(\lambda) \circ (IX(\lambda - 1)) = I = -(IX(\lambda - 1)) \circ T(\lambda) + T(\lambda) \circ (X(\lambda + 1))$ as endomorphisms of $E(\lambda - 1)E(\lambda + 1)$.
- For $r \geq 0$, we have $\text{Hom}(E^{(r)}(\lambda), E^{(r)}(\lambda)\langle i \rangle) = 0$ if $i < 0$ and $\text{End}(E^{(r)}(\lambda)) = \mathbb{k} \cdot I$.

Given a categorical \mathfrak{sl}_2 action, for each $\lambda \geq 0$ we may construct the Rickard complex [CKL2]

$$\Theta_* : \mathcal{D}(\lambda) \rightarrow \mathcal{D}(-\lambda).$$

The terms in the complex are

$$\Theta_s = F^{(\lambda+s)}(s)E^{(s)}(\lambda + s)\langle -s \rangle,$$

where $s = 0, \dots, (N - \lambda)/2$. The differential $d_s : \Theta_s \rightarrow \Theta_{s-1}$ is given by the composition of maps

$$F^{(\lambda+s)}E^{(s)}\langle -s \rangle \xrightarrow{\iota\iota} F^{(\lambda+s-1)}FEE^{(s-1)}\langle -(\lambda + s - 1) - (s - 1) - s \rangle \xrightarrow{\epsilon} F^{(\lambda+s-1)}E^{(s-1)}\langle -s + 1 \rangle.$$

Then we have the following theorem, proved in [CKL2].

Theorem 3.1. *Suppose the underlying weight space categories $\mathcal{D}(\lambda)$ are triangulated. Then complex Θ_* has a unique convolution \mathbb{T} , and $\mathbb{T} : \mathcal{D}(-\lambda) \rightarrow \mathcal{D}(\lambda)$ is an equivalence of triangulated categories.*

3.2. A Geometric Example. The basic example of a categorical \mathfrak{sl}_2 action comes from Grassmanian geometry, and we refer to [CKL2] for complete details.

Fix $N > 0$. For our weight spaces we will take the derived category of coherent sheaves on the cotangent bundle to the Grassmannian $T^*\mathbb{G}(k, N)$. We use shorthand $Y(\lambda) = T^*\mathbb{G}(k, N)$, where $k = (N - \lambda)/2$. These spaces have a particularly nice geometric description,

$$T^*\mathbb{G}(k, N) \cong \{(X, V) : X \in \text{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } \mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0\},$$

where $\text{End}(\mathbb{C}^N)$ denotes the space of complex $N \times N$ matrices. (The notation $\mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0$ means that $X(\mathbb{C}^N) \subset V$ and that $X(V) = 0$.) Forgetting X corresponds to the projection $T^*\mathbb{G}(k, N) \rightarrow \mathbb{G}(k, N)$ while forgetting V gives a resolution of the variety

$$\{X \in \text{End}(\mathbb{C}^N) : X^2 = 0 \text{ and } \text{rank}(X) \leq \min(k, N - k)\}.$$

On $T^*\mathbb{G}(k, N)$ we have the tautological rank k vector bundle V as well as the quotient \mathbb{C}^N/V .

To describe the kernels \mathcal{E} and \mathcal{F} we will need the correspondences

$$W^r(\lambda) \subset T^*\mathbb{G}(k + r/2, N) \times T^*\mathbb{G}(k - r/2, N)$$

defined by

$$\begin{aligned} W^r(\lambda) := \{(X, V, V') : X \in \text{End}(\mathbb{C}^N), \dim(V) = k + \frac{r}{2}, \dim(V') = k - \frac{r}{2}, \\ 0 \subset V' \subset V \subset \mathbb{C}^N, \mathbb{C}^N \xrightarrow{X} V', \text{ and } V \xrightarrow{X} 0\}. \end{aligned}$$

(Here, as before, λ and k are related by the equation $k = (N - \lambda)/2$.)

There are two natural projections $\pi_1 : (X, V, V') \mapsto (X, V)$ and $\pi_2 : (X, V, V') \mapsto (X, V')$ from $W^r(\lambda)$ to $Y(\lambda - r)$ and $Y(\lambda + r)$ respectively. Together they give us an embedding

$$(\pi_1, \pi_2) : W^r(\lambda) \subset Y(\lambda - r) \times Y(\lambda + r).$$

On $W^r(\lambda)$ we have two natural tautological bundles, namely $V := \pi_1^*(V)$ and $V' := \pi_2^*(V)$, where the prime on the V' indicates that the vector bundle is the pullback of the tautological bundle by the second projection. We also have natural inclusions

$$0 \subset V' \subset V \subset \mathbb{C}^N \cong \mathcal{O}_{W^r(\lambda)}^{\oplus N}.$$

We now define the kernel $\mathcal{E}^{(r)}(\lambda) \in D(Y(\lambda - r) \times Y(\lambda + r))$ by

$$\mathcal{E}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(\mathbb{C}^N/V')^{-r} \det(V)^r \left\{ \frac{r(N - \lambda - r)}{2} \right\}.$$

Similarly, the kernel $\mathcal{F}^{(r)}(\lambda) \in D(Y(\lambda + r) \times Y(\lambda - r))$ is defined by

$$\mathcal{F}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(V'/V)^{\lambda} \left\{ \frac{r(N + \lambda - r)}{2} \right\}.$$

These kernels define functors (Fourier-Mukai transforms) $E^{(k)}$ and $F^{(k)}$, and in [CKL2] we define natural transformations which enhance these functors to a full categorical \mathfrak{sl}_2 action.

As a result, we may define the Rickard complex Θ . Convolution with this complex gives new equivalences of triangulated categories between categories corresponding to opposite \mathfrak{sl}_2 weight spaces.

Corollary 3.2. [CKL3] *The complex Θ defines an equivalence between derived categories of coherent sheaves of cotangent bundles to dual Grassmanians*

$$\Theta : D(T^*(G(k, N))) \simeq D(T^*(G(N - k, N))).$$

4. FURTHER DEVELOPEMENTS

The notion of \mathfrak{sl}_2 -categorification goes back at least to the paper [BFK], which inspired much of the subsequent work on algebraic aspects of categorification. After the seminal contribution [CR], which contains several algebraic examples of \mathfrak{sl}_2 categorifications, various geometric aspects of categorical \mathfrak{sl}_2 representation theory were developed in [CKL1], [CKL2], [CKL3].

On the other hand, it is quite natural to categorify the entire quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, rather than just the finite dimensional representations. This has been accomplished by Rouquier [R], and by Lauda [L]. Moreover, the entire story can be generalized and repeated, with the lead actor \mathfrak{sl}_2 replaced by an arbitrary symmetrizable Kac-Moody Lie algebra \mathfrak{g} . This is the subject of the significant work of Khovanov-Lauda [KL] and, independently, Rouquier [R].

REFERENCES

- [BFK] J. Bernstein, I. Frenkel, and M. Khovanov, A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors, *Selecta Math.* (5) (1999), 199–241; [math.QA/0002087](#).
- [CR] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification, *Ann. of Math.* **167** (2008), no. 1, 245–298; [math.RT/0407205](#).
- [CKL1] S. Cautis, J. Kamnitzer and A. Licata, Categorical geometric skew Howe duality, to appear.
- [CKL2] S. Cautis, J. Kamnitzer and A. Licata, Derived equivalences for cotangent bundles of Grassmannians via categorical \mathfrak{sl}_2 actions, to appear.
- [CKL3] S. Cautis, J. Kamnitzer and A. Licata, Coherent Sheaves and Categorical \mathfrak{sl}_2 Actions, to appear.
- [KL] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, II, and III; [math.QA/0803.4121](#), [math.QA/0804.2080](#), and [math.QA/0807.3250](#).
- [L] A. Lauda, A categorification of quantum \mathfrak{sl}_2 , [arXiv:0803.3652v2](#).
- [R] R. Rouquier, 2-Kac-Moody algebras; [math.RT/0812.5023](#).