

# The ternary Goldbach problem

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# The ternary Goldbach problem: what is it?

## What was known?

**Ternary Golbach conjecture (1742), or three-prime problem:**

Every odd number  $n \geq 7$  is the sum of three primes.

(Binary Goldbach conjecture:  
every even number  $n \geq 4$  is the sum of two primes.)

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## Introduction

## The circle method

## The major arcs

## Minor arcs

## Conclusion

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**Hardy-Littlewood (1923):** There is a  $C$  such that every odd number  $\geq C$  is the sum of three primes, if we assume the generalized Riemann hypothesis (GRH).

**Vinogradov (1937):** The same result, unconditionally.

# Bounds for more prime summands

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We also know:

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and after intermediate results by **Klimov (1969)**

( $K = 6 \cdot 10^9$ ), **Klimov-Piltay-Sheptiskaya, Vaughan, Deshouillers (1973), Riesel-Vaughan...**,

every even  $n \geq 2$  is the sum of  $\leq 6$  primes (**Ramaré, 1995**)

every odd  $n > 1$  is the sum of  $\leq 5$  primes (**Tao, 2012**).

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Ternary Goldbach holds for all  $n$  conditionally on the  
generalized Riemann hypothesis (GRH)  
(**Deshouillers-Effinger-te Riele-Zinoviev, 1997**)

# Bounds for ternary Goldbach

Every odd  $n \geq C$  is the sum of three primes (Vinogradov)

Bounds for  $C$ ?  $C = 3^{3^{15}}$  (Borodzin, 1939),  
 $C = 3.33 \cdot 10^{43000}$  (Wang-Chen, 1989),  $C = 2 \cdot 10^{1346}$   
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Verification for small  $n$ :

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taken together with results by Ramaré-Saouter and Platt,  
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We must diminish  $C$  from  $2 \cdot 10^{1346}$  to  $\sim 10^{30}$ .

# Exponential sums and the circle method

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where

$\eta(t) = e^{-t}$  (Hardy-Littlewood),  $\eta(t) = 1_{[0,1]}$  (Vinogradov),  
 $\Lambda(n) = \log p$  if  $n = p^{\alpha}$ ,  $\Lambda(n) = 0$  if  $n$  is not a prime power  
(**von Mangoldt** function)

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The crucial identity:

$$\begin{aligned} & \sum_{n_1+n_2+n_3=N} \Lambda(n_1) \Lambda(n_2) \Lambda(n_3) \eta(n_1/x) \eta(n_2/x) \eta(n_3/x) \\ &= \int_{\mathbb{R}/\mathbb{Z}} (S_{\eta}(\alpha, x))^3 e(-N\alpha) d\alpha. \end{aligned}$$

We must show that this integral is  $> 0$ .

# Major and minor arcs

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$m_{a,q} \subset (a/q - 1/qQ, a/q + 1/qQ)$  around  $a/q$ ,  $q \leq Q$ ,  
where  $Q \leq x$ . (**Farey fractions**)

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In general, up to now,  $m(x) \sim (\log x)^k$ ,  $k > 0$  constant.

Let  $\mathfrak{M}$  be the union of major arcs and  $\mathfrak{m}$  the union of  
minor arcs.

We want to estimate  $\int_{\mathfrak{M}} (S_{\eta}(\alpha, x))^3 e(-N\alpha) d\alpha$  and bound  
 $\int_{\mathfrak{m}} |S_{\eta}(\alpha, x)|^3 d\alpha$  from above.

# The major arcs

To estimate  $\int_{\mathfrak{M}} (S_{\eta}(\alpha, x))^3 e(-N\alpha)$ , we need to estimate  $S_{\eta}(\alpha, x)$  for  $\alpha$  near  $a/q$ ,  $q$  small ( $q \leq m(x)$ ).

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We do this studying  $L(s, \chi)$  for Dirichlet characters mod  $q$ , i.e., characters  $\chi : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}$ .

$$L(s, \chi) := \sum_n \chi(n) n^{-s}$$

for  $\Re(s) > 1$ ; this has an analytic continuation to all of  $\mathbb{C}$  (with a pole at  $s = 1$  if  $\chi$  is trivial).

We express  $S_{\eta}(\alpha, x)$ ,  $\alpha = a/q + \delta/x$ , as a sum of

$$S_{\eta, \chi}(\delta/x, x) = \sum_{n=1}^{\infty} \Lambda(n) \chi(n) e(\delta n/x) \eta(n/x)$$

for  $\chi$  varying among all Dirichlet characters modulo  $q$ .

# The explicit formula

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- (a) the term  $F_{\delta}(1)x$  appears only for  $\chi$  principal ( $\sim$  trivial),
- (b)  $\rho$  runs over the complex numbers  $\rho$  with  $L(\rho, \chi) = 0$  and  $0 < \Re(\rho) \leq 1$  (called “non-trivial zeroes”),
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Analytic on a strip  $x_0 < \Re(s) < x_1$  in  $\mathbb{C}$ .

It is a Laplace transform (or Fourier transform!) after a change of variables.



# Where are the zeroes of $L(s, \chi)$ ?

Let  $\rho = \sigma + it$  be any non-trivial zero of  $L(s, \chi)$ .

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## What we can also know:

for a given  $\chi$ , we can verify GRH for  $L(s, \chi)$  “up to a height  $T_0$ ”. This means: verify that every zero  $\rho$  with  $|\Im(\rho)| \leq T_0$  satisfies  $\sigma = 1/2$ .

## Verifying GRH up to a given height

For the purpose of proving strong bounds that solve ternary Goldbach, **zero-free regions** are far too weak. We must rely on **verifying** GRH for several  $L(s, \chi)$ ,  $|t| \leq T_0$ .

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For  $\chi$  trivial ( $\chi(x) = 1$ ),  $L(s, \chi) = \zeta(s)$ .

The Riemann hypothesis has been verified up to  
 $|t| \leq 2.4 \cdot 10^{11}$  (**Wedeniowski** 2003),  $|t| \leq 1.1 \cdot 10^{11}$  (**Platt** 2012, **rigorous**),  $|t| \leq 2.4 \cdot 10^{12}$  (**Gourdon-Demichel** 2004, not duplicated to date).

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This has been extended up to  $q \leq 2 \cdot 10^5$ ,  $q$  odd, and  
 $q \leq 4 \cdot 10^5$ ,  $q$  pair ( $|t| \leq 200 + 7.5 \cdot 10^7/q$ ) (**Platt** 2013).



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We must choose  $\eta$  so that

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For  $\eta(t) = e^{-t}$ , the Mellin transform of  $\eta(t)e(\delta t)$  is

$$F_{\delta}(s) = \frac{\Gamma(s)}{(1 - 2\pi i \delta)^s}.$$

Decreases as  $e^{-\lambda|\tau|}$ ,  $\lambda = \tan^{-1} \frac{1}{2\pi|\delta|}$ , for  $s = \sigma + i\tau$ ,  $|\tau| \rightarrow \infty$ . If  $\delta \gg 1$ , then  $\lambda \sim \frac{1}{2\pi|\delta|}$ . Problem:  $e^{-|\tau|/2\pi\delta}$  does not decay very fast for  $\delta$  large!

# The Gaussian smoothing

Instead, we choose  $\eta(t) = e^{-t^2/2}$ . The Mellin transform  $F_\delta$  is then a **parabolic cylinder function**.

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The main term in  $F_\delta(\sigma + i\tau)$  behaves as

$$e^{-\frac{\pi}{4}|\tau|}$$

for  $\delta$  small,  $\tau \rightarrow \pm\infty$ , and as

$$e^{-\frac{1}{2}\left(\frac{|\tau|}{2\pi\delta}\right)^2}$$

for  $\delta$  large,  $\tau \rightarrow \pm\infty$ .

# Major arcs: conclusions

Thus we obtain estimates for  $S_{\eta, \chi}(\delta/x, x)$ , where

$$\eta(t) = g(t)e^{-t^2/2},$$

and  $g$  is any “band-limited” function:

$$g(t) = \int_{-R}^R h(r)t^{-ir}dr$$

where  $h : [-R, R] \rightarrow \mathbb{C}$ .

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All the rest of the circle must be minor arcs;  $m(x)$  must be a constant  $M$ . (Writer for *Science*: “Muenster cheese” rather than “Swiss cheese”.)

Thus, minor-arc bounds will have to be very strong.

# Back to the circle

We use two functions  $\eta, \eta_*$  instead of a function  $\eta$ .

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$$\int_{\mathfrak{m}} |S_{\eta}(\alpha, x)|^2 |S_{\eta_*}(\alpha, x)| d\alpha \leq \max_{\alpha \in \mathfrak{m}} |S_{\eta_*}(\alpha, x)| \cdot L_2, \quad (1)$$

where  $L_2 = \int_{\mathfrak{m}} |S_{\eta}(\alpha, x)|_2^2 d\alpha$ .

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We use two functions  $\eta, \eta_*$  instead of a function  $\eta$ . It is trivial that

$$\int_{\mathfrak{m}} |S_{\eta}(\alpha, x)|^2 |S_{\eta_*}(\alpha, x)| d\alpha \leq \max_{\alpha \in \mathfrak{m}} |S_{\eta_*}(\alpha, x)| \cdot L_2, \quad (1)$$

where  $L_2 = \int_{\mathfrak{m}} |S_{\eta}(\alpha, x)|^2 d\alpha$ . Bounding  $L_2$  is easy ( $\sim x \log x$  by Plancherel).

We must bound  $|S_{\eta_*}(\alpha)|$ ,  $\alpha \sim a/q + \delta/x$ ,  $q > M$ .

**It is possible to improve (1): Heath-Brown** replaces  $x \log x$  by  $2e^{\gamma} x \log q$ . A new approach based on **Ramaré's** version of the large sieve (cf. *Selberg*) replaces this by  $2x \log q$ .

The idea is that one can give good bounds for the integral over the arcs with denominator between  $r_0$  and  $r_1$  (say).

# What weight $\eta_+$ ?

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The main term for the number of (weighted) solutions to  $N = p_1 + p_2 + p_3$  will be proportional to

$$\int_0^\infty \int_0^\infty \eta_+(t_1) \eta_+(t_2) \eta_* \left( \frac{N}{x} - t_1 - t_2 \right) dt_1 dt_2, \quad (2)$$

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To maximize (2) (divided by  $|\eta_+|^2 |\eta_*|_\infty$ ), define  $\eta_+(t)$  so that (a) it is approximately symmetric around  $t = 1$ , (b) it is (almost) supported on  $[0, 2]$ .

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Solution: since  $\eta(t) = g(t)e^{-t^2/2}$ , we let  $g$  be a band-limited approximation to  $e^t \cdot I_{[0,2]}$ .

## What weight $\eta_*$ ?

In order to estimate  $S_{\eta_*}$  on the major arcs, we want a  $\eta_*$  whose Mellin transform **decreases exponentially** for  $\Re(s)$  bounded,  $\Im(s) \rightarrow \pm\infty$ .

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Vinogradov chose  $\eta_* = 1_{[0,1]}$ .

We would like:  $\eta_+(x) = f *_M f$ , where

$$(f *_M f)(t_0) = \int_0^\infty f(t) f\left(\frac{t_0}{t}\right) \frac{dt}{t},$$

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**Solution:**  $\eta_* := \eta_0 *_M f *_M f$ , where  $\eta_0$  has a Mellin transform with exponential decay.

If we know  $S_{f*f}(\alpha, x)$  **or**  $S_{\eta_0}(\alpha, x)$ , we know  $S_{\eta_*}(\alpha, x)$ .

# The new bound for minor arcs

## Theorem (Helfgott, May 2012 – March 2013)

Let  $x \geq x_0$ ,  $x_0 = 2.16 \cdot 10^{20}$ . Let  $2\alpha = a/q + \delta/x$ ,  $\gcd(a, q) = 1$ ,  $|\delta/x| \leq 1/qQ$ , where  $Q = (3/4)x^{2/3}$ . If  $q \leq x^{1/3}/6$ , then  $|S_{\eta_2}(\alpha, x)|/x$  is less than

$$\begin{aligned} & \frac{R_{x, \delta_0 q}(\log \delta_0 q + 0.002) + 0.5}{\sqrt{\delta_0 \phi(q)}} + \frac{2.491}{\sqrt{\delta_0 q}} \\ & + \frac{2}{\delta_0 q} \min \left( \frac{q}{\phi(q)} \left( \log \delta_0^{7/4} q^{13/4} + \frac{80}{9} \right), \frac{5}{6} \log x + \frac{50}{9} \right) \\ & + \frac{2}{\delta_0 q} \left( \log q^{\frac{80}{9}} \delta_0^{\frac{16}{9}} + \frac{111}{5} \right) + 3.2x^{-1/6}, \end{aligned}$$

where  $\delta_0 = \max(2, |\delta|/4)$ ,

$$R_{x, t_1, t_2} = 0.4141 + 0.2713 \log \left( 1 + \frac{\log 4t_1}{2 \log \frac{9x^{1/3}}{2.004t_2}} \right).$$



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Theorem (Helfgott, May 2012 – March 2013, bound for  $q$  large)

If  $q > x^{1/3}/6$ , then

$$|S_\eta(\alpha, x)| \leq 0.27266x^{5/6}(\log x)^{3/2} + 1217.35x^{2/3} \log x.$$

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For  $x = 10^{25}$ ,  $q \sim 1.5 \cdot 10^5$ ,  $|\delta| < 8$  (the most delicate case)

$$R_{x, \delta_0 q} = 0.5833 \dots$$

## Worst-case comparison

Let us compare the results here (2012-2013) with those of Tao (Feb 2012) for  $q$  highly composite,  $|\delta| < 8$ :

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$q_0$	$\frac{ S_\eta(a/q, x) }{x}, \text{ HH}$	$\frac{ S_\eta(a/q, x) }{x}, \text{ Tao}$
$10^5$	0.04521	0.34475
$1.5 \cdot 10^5$	0.03820	0.28836
$2.5 \cdot 10^5$	0.03096	0.23194
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**Table:** Upper bounds on  $x^{-1}|S_\eta(a/2q, x)|$  for  $q \geq q_0$ ,  $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 | q$ ,  $|\delta| \leq 8$ ,  $x = 10^{25}$ . The trivial bound is 1.

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Meaning: GRH verification needed only for  $q \leq 1.5 \cdot 10^5$ ,  $q$  odd, and  $q \leq 3 \cdot 10^5$ ,  $q$  even.

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Qualitative improvements:

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Qualitative improvements:

- cancellation within Vaughan's identity
- $\delta/x = \alpha - a/q$  is a friend, not an enemy:



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## Qualitative improvements:

- cancellation within Vaughan's identity
- $\delta/x = \alpha - a/q$  is a friend, not an enemy:
  - In type I: (a) decrease of  $\widehat{\eta}$ ,  
change in approximations;  
In type II: scattered input to the large sieve
- relation between the circle method and the large sieve – in its version for primes;
- the benefits of a continuous  $\eta$  (also in Tao, Ramaré),

# Cancellation within Vaughan's identity

Vaughan's identity:

$$\Lambda = \mu_{\leq U} * \log - \Lambda_{\leq V} * \mu_{\leq U} * 1 + 1 * \mu_{> U} * \Lambda_{> V} + \Lambda_{\leq V},$$

where  $f_{\leq V}(n) = f(n)$  if  $n \leq V$ ,  $f_{\leq V}(n) = 0$  if  $n > V$ . (Four summands: type I, type I, type II, negligible.)

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- Advantage: flexibility – we may choose  $U$  and  $V$ ;
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Alternative would have been: use a log-free formula (e.g. Daboussi-Rivat); proceeding as above seems better in practice.

# How to recover factors of log

In type I sums:

We use cancellation in  $\sum_{n \leq M: d|n} \mu(n)/n$ .

**This is allowed:** we are using only  $\zeta$ , not  $L$ .

**This is explicit:** **Granville-Ramaré, El Marraki, Ramaré.**

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In type II sums:

Proof of cancellation in  $\sum_{m \leq M} (\sum_{d > U} \mu(d))^2$ , even for  $U$  almost as large as  $M$ .



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Application of the large sieve for primes.

# The “error” $\delta/x = \alpha - a/q$ is a friend

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In type II:

- $\widehat{\eta}(\delta) \ll 1/\delta^2$  (so that  $|\eta''|_1 < \infty$ ),
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In type II: the angles  $m\alpha$  are separated by  $\geq \delta/x$  (**even when  $m \geq q$** ). We can apply the large sieve to *all*  $m\alpha$  in one go. We can even use prime support: double scattering, by  $\delta$  and by **Montgomery's** lemma.

# Final result

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All goes well for  $n \geq 10^{30}$  (or well beneath that). As we have seen, the case  $n \leq 10^{30}$  is already done (computation).

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## Theorem (Helfgott, May 2013)

*Every odd number  $n \geq 7$  is the sum of three prime numbers.*