Loop Groups and Characteristic Classes

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Characteristic Classes and Chern-Weil Theory

- $Q \rightarrow M$ a $G$-bundle
- A characteristic class is a cohomology class $c(Q) \in H^*(M)$ which is natural with respect to pull-backs:

$$f^* c(Q) = c(f^* Q)$$

- All characteristic classes are pulled-back from $H^*(BG)$:

$$c(Q) = f_Q^* c(EG)$$

- $G$ compact, connected $\Rightarrow H^{2k}(BG) \cong S^k(g^*)^G$

$$H^{2k}(BG) \ni f \mapsto f(F, \ldots, F) \in H^{2k}(M)$$
There exists a bijection on isomorphism classes:

\[ G \to \tilde{P} \quad \text{and} \quad LG \to P \]

\[ M \times S^1 \]

\[ P_m = \Gamma(\tilde{P}|_\{m\} \times S^1) \]

\[ \tilde{P} = \frac{P \times G \times S^1}{LG} \]
Caloron Correspondence

On the level of connections:

\[(\tilde{P}, \tilde{A}) \quad \leftrightarrow \quad (P, (A, \Phi))\]

- \(\Phi \sim S^1\) part of \(\tilde{A}\)

i.e. \[\tilde{P}|_{S^1} \simeq G \times S^1 \quad \Rightarrow \quad \tilde{A}|_{S^1} \sim \Theta + \Phi \, d\theta\]

- \(\Phi : P \to L_g\) satisfying

\[\Phi(p \gamma) = \text{ad}(\gamma^{-1})\Phi(p) + \gamma^{-1} \partial \gamma\]

- \(\tilde{A} = \text{ad}(g^{-1})A(\theta) + \Theta + \text{ad}(g^{-1})\Phi \, d\theta\)
String Classes

For $P \xrightarrow{LG} M$, define:

$$H^{2k}(BG) \xrightarrow{\text{cw}(\tilde{P})} H^{2k}(M \times S^1) \xrightarrow{\int_{S^1}} H^{2k-1}(M)$$

$$f \xrightarrow{} f(\tilde{F}^k) \xrightarrow{} \int_{S^1} f(\tilde{F}^k)$$

- We have: $\tilde{F} = \text{ad}(g^{-1}) (F + \nabla \Phi \, d\theta)$
  
  (where $\nabla \Phi = d\Phi + [A, \Phi] - \partial A$)

$$\Rightarrow \int_{S^1} f(\tilde{F}^k) = k \int_{S^1} f(\nabla \Phi, F^{k-1}) \, d\theta$$
String Classes

**Proposition**

\[
k \int_{S^1} f(\nabla \Phi, F^{k-1}) \, d\theta \text{ is:}
\]

- closed
- independent of choice of A and \( \Phi \)
- natural

We call

\[
s_f(P) = k \int_{S^1} f(\nabla \Phi, F^{k-1}) \, d\theta \in H^{2k-1}(M)
\]

the string class of \( P \) associated to \( f \)
Example (Murray–Stevenson (2003))

\[ k = 2, \ f = p_1 \in H^4(BG) : \]

\[ s_{p_1}(P) = \frac{-1}{4\pi^2} \int_{S^1} \langle \nabla \Phi, F \rangle \, d\theta \in H^3(M) \]

- Obstruction to lifting \( P \xrightarrow{LG} M \) to \( \hat{P} \xrightarrow{\hat{L}G} M \)
Loop Groups and Classifying Spaces

Look at

1. $\Omega G$
   (difficult caloron correspondence, easy classifying theory)
2. $LG$
   (easy caloron correspondence, difficult classifying theory)
String Classes for $\Omega G$-bundles

Universal $\Omega G$-bundle (Carey–Mickelsson (2000)):

- path fibration $PG \to G$

Classifying map:

- Solve $\Phi(p) = g^{-1} \partial g$
- $g = \text{hol}_\Phi(p)$ – Higgs field holonomy
Choose a connection and Higgs field for \( PG \rightarrow G \)

**Proposition**

\[
    s_f(PG) = \left( -\frac{1}{2} \right)^{k-1} \frac{k!(k - 1)!}{(2k - 1)!} f(\Theta, [\Theta, \Theta]^{k-1})
\]

That is

\[
    s_f(PG) = \tau(f),
\]

where \( \tau : H^{2k}(BG) \rightarrow H^{2k-1}(G) \) is the transgression map.
Theorem (Murray–V (2010))

If $P \to M$ is an $\Omega G$-bundle and

$$s(P) : S^k(g^*)^G \to H^{2k-1}(M)$$

is the map which associates to any invariant polynomial $f$ the string class of $P$ (i.e. $s(P)(f) = s_f(P)$), then the following diagram commutes
String Classes for $LG$-bundles

$P \to M$ an $LG$-bundle

- Want a model for $ELG \to BLG$ and a map $M \to BLG$

$LG \cong \Omega G \rtimes G$ therefore take

$$ELG = PG \times EG$$

$$BLG = G \times_G EG$$

So

$$H(BLG) = H(G \times_G EG) = H_G(G)$$

- Classifying map: $c_{LG} = (\text{hol}_\Phi, c_G)$
String Classes for $LG$-bundles

So we expect

$$H^{2k}(BG) \xrightarrow{cw(\tilde{P})} H^{2k}(M \times S^1)$$

$$\xrightarrow{?} s(P)$$

$$\xrightarrow{c^*_L} H^{2k-1}(G) \xrightarrow{c^*_L} H^{2k-1}(M)$$

Therefore we must calculate $s_f(ELG)$
Equivariant Cohomology

- $G$ acts on $X$

Want to study $H(X/G)$

- Borel model:

$$H_G(X) = H(X \times_G EG)$$

- Cartan model:

$$\Omega_G(X) = \left( S(g^*) \otimes \Omega(X) \right)^G$$

$$(d_G \omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi))$$

$$H_G(X) = H(\Omega_G(X), d_G)$$

Borel model $\simeq$ Cartan model — Mathai–Quillen isomorphism
Equivariant Transgression Forms

Recall: \[ \tau(f) = \left( -\frac{1}{2} \right)^{k-1} \frac{k!(k-1)!}{(2k-1)!} f(\Theta, [\Theta, \Theta]^{k-1}) \]

\[ = - \int_0^1 f(F_t^k) \]

where \( F_t = F(t\Theta) = \frac{1}{2} t^2 [\Theta, \Theta] \)

Define (Jeffrey (1995), Alekseev–Meinrenken (2009)):

\[ \tau_G(f) = - \int_0^1 f \left( (F_G(t\Theta)(\xi) + \xi)^k \right) \]

where \( F_G(t\Theta) = d_G(t\Theta) + \frac{1}{2} t^2 [\Theta, \Theta] \)
Universal String Class

Calculate \( s_f(ELG) \):

- Choose \( A \) and \( \Phi \)
- Calculate \( F \) and \( \nabla \Phi \)
- Calculate \( k \int_{S^1} f(\nabla \Phi, F^{k-1}) \, d\theta \in H^{2k-1}(G \times G \text{EG}) \)
- Apply Mathai–Quillen isomorphism

\[
H^{2k-1}(G \times G \text{EG}) \xrightarrow{\sim} H^{2k-1}(\Omega_G(G))
\]

**Proposition**

\[
s_f(ELG) = \tau_G(f)
\]
String Classes for $LG$-bundles

Theorem (V (arXiv:1005.4243))

If $P \rightarrow M$ is an $LG$-bundle and

$$s(P): S^k(g^*)^G \rightarrow H^{2k-1}(M)$$

is the map which associates to any invariant polynomial $f$ the string class of $P$ (i.e. $s(P)(f) = s_f(P)$), then the following diagram commutes:

$$
\begin{array}{cccccc}
H^{2k}(BG) & \xrightarrow{cw(\tilde{P})} & H^{2k}(M \times S^1) \\
\downarrow \tau_G & & \downarrow \int_{S^1} \\
H^{2k-1}_G(G) & \xrightarrow{c^*_LG} & H^{2k-1}(M)
\end{array}
$$