Families of Dirac operators and quantum affine groups

Deforming twisted K-theory

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References

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Outline of the talk

- Background: Twisted K-theory from Dirac type operators on loop groups
- q-Deformation of the Dirac family
- The q-fermionic algebra and generalized affine Hecke algebra
- Quantum adjoint module
- Twisting and the central element in $U_q(\hat{\mathfrak{g}})$

Gerbes and Fredholm operators

X is a topological parameter space, $Fred_*$ the space of self-adjoint Fredholm operators in a complex Hilbert space H with both positive and negative essential spectrum. This is a universal classifying space for K^1. Actually, one can take as the **definition:**

$$K^1(X) = \{\text{homotopy classes of maps } f : X \to Fred_*\}$$

Without loss of generality we can require that the Fredholm operators have a discrete spectrum. In the even case

$$K^0(X) = \{\text{homotopy classes of maps } f : X \to Fred\}$$

where $Fred$ is the space of all Fredholm operators in H.

The Dixmier-Douady class

The Chern character

$$ch : K^1(X) \to H^{odd}(X, \mathbf{Z})$$

is an additive map to odd cohomology classes. In particular, the degree 3 component $DD(f) = ch_3(f)$ of $[f] \in K^1(X)$ is called the **Dixmier-Douady class** of the gerbe defined by the the family $f(x)$ of Fredholm operators. In the de Rham cohomology an equivalent construction of $DD(f)$ comes from the family $L_{\lambda\lambda'}$ of complex line bundles. One can choose the curvature forms $\omega_{\lambda\lambda'}$ such that

$$\omega_{\lambda\lambda'} + \omega_{\lambda'\lambda''} = \omega_{\lambda\lambda''}$$

and with a partition of unity $\sum \rho_\lambda = 1$ subordinate to the cover by the open sets U_λ one has

$$DD(f) = \sum_\lambda d\rho_\lambda \wedge \omega_{\lambda\lambda'}$$

and this does not depend on the choice of λ'.

Twisting K-theory with the D-D class

The previous example can generalized: Let $P \to X$ be a principal bundle with a right action of a group $\mathcal{G}$. Fix a cocycle

$$\omega : P \times \mathcal{G} \to PU(H), \text{ with } \theta(p; g_1 g_2) = \theta(p; g_1)\theta(pg_1; g_2)$$

where $PU(H) = U(H)/S^1$. Then a map $f : P \to Fred(H)$ with

$$f(pg) = \theta(p; g)^{-1} f(p)\theta(p; g)$$

defines an element in the twisted K-group $K^0(X; \theta)$. The group K^1 is defined similarly using $Fred_*$ instead of $Fred$.
The groups $K^*(X, \theta)$ actually depend only on the class of the $PU(H)$ bundle defined by the cocycle θ. This class is the Dixmier-Douady class in $H^3(X, \mathbf{Z})$.

Example: The WZW model

Families of Dirac operators D_A transform covariantly under the (projective) gauge group action, defining an element in $K^*(\mathcal{A}/\mathcal{G}, \theta)$, where θ is defined by the projective action $g \mapsto \hat{g}$ in the Fock spaces? **False:** The quantized Dirac operators are essentially positive, we need operators with both negative and positive essential spectrum. Solution: Hamiltonians in supersymmetric WZW model:

$$\begin{aligned} Q_A &= i\psi_a^n T_a^{-n} + \frac{i}{12}\lambda_{abc}\psi_a^n\psi_b^m\psi_c^{-n-m} + i(k+\kappa)\psi_a^n A_a^{-n} \\ \psi_a^n\psi_b^m &+ \psi_b^m\psi_a^n = 2\delta_{ab}\delta_{n,-m} \\ [T_a^n, T_b^m] &= \lambda_{abc}T_c^{n+m} + k\delta_{ab}n\delta_{n,-m}. \end{aligned}$$

Here A_a^n's are the Fourier components of a vector potential on the circle.

The family Q_A transforms covariantly under the projective representation of level $k+\kappa$ the loop group $\mathcal{G} = LG$ defining an element in $K(G, k+\kappa)$ corresponding to the D-D class $[H]$ in $H^3(G, \mathbf{Z})$ equal to $k+\kappa$ times the basic class in $H^3(G) = \mathbf{Z}$ when G is a simple simply connected compact Lie group. Actually, since $\mathcal{A}/\Omega G = G$ and $G \subset LG$, we have an G equivariant class, element of $K^*_G(G, H)$.

Morally, the family Q_A is a family of Dirac operators on the loop group LG, coupled to a gauge connection A on a complex line bundle over LG.

Quantum affine algebra

$\mathfrak{g}$ a simple finite-dimensional Lie algebra, $\hat{\mathfrak{g}}$ the associated affine Lie algebra. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is generated by $e_0, e_1, \ldots, e_\ell, f_0, f_1, \ldots, f_\ell, K_0, K_1, \ldots, K_\ell, K_0^{-1}, \ldots, K_\ell^{-1}$ with the relations

$$
\begin{aligned}
[e_i, f_i] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, K_i K_j = K_j K_i \\
K_i e_j K_i^{-1} &= q^{\alpha_{ij}} e_j, K_i f_j K_i^{-1} = q^{-\alpha_{ij}} f_j \\
\sum_{k=0}^{1-a_{ij}} & (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_q e_i^{1-a_{ij}-k} e_j e_i^k = 0 (i \neq j) \\
\sum_{k=0}^{1-a_{ij}} & (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_q f_i^{1-a_{ij}-k} f_j f_i^k = 0 (i \neq j)
\end{aligned}
$$

with

$$\left[\begin{array}{c} m \\ k \end{array} \right]_q = \frac{m_q(m-1)_q \dots (m-k+1)_q}{k_q(k-1)_q \dots 1_q}$$

$$k_q = 1 + q + \dots q^{k-1}$$

q is a positive real number in this talk and the integers a_{ij} are the matrix elements of the Cartan matrix of $\hat{\mathfrak{g}}$.

The Dirac operator

Let A_i^n with $n \in \mathbf{Z}$ and $i = 0, 1, \ldots dim\,\mathfrak{g}$ be a basis for the q-affine adjoint module. Under $\mathfrak{g}$ each 'Fourier mode' A^n transforms acording to the adjoint representation of $U_q(\mathfrak{g})$, which is a q-deformation of the adjoint representation of $\mathfrak{g}$. The generator e_0 increases the index n by one unit, f_0 decreases it by one unit. For example, for $\mathfrak{g} = \mathfrak{sl}(2)$ one has the explicit formulas

$$
\begin{aligned}
e_1 A_1^n &= f_0 A_1^n = 0, f_1 A_1^n = A_0^n, e_0 A_1^n = A_0^{n+1} \\
e_1 A_0^n &= (q + q^{-1}) A_1^n, f_0 A_0^n = (q + q^{-1}) A_1^{n-1} \\
f_1 A_0^n &= A_{-1}^n, e_0 A_0^n = A_{-1}^{n+1} \\
e_1 A_{-1}^n &= (q + q^{-1}) A_0^n, f_0 A_{-1}^n = (q + q^{-1}) A_0^{n-1}, f_1 A_{-1}^n = 0 = e_0 A_{-1}^n \\
K_1 A_i^n &= q^{2i} A_i^n = K_0^{-1} A_i^n.
\end{aligned}
$$

The Dirac operator

The vectors A_i^n will be constructed as operators acting in a Fock space carrying a representation of $U_q(\hat{\mathfrak{g}})$ such that the adjoint action is given by

$$x.A_i^n = \sum_{(x)} x' A_i^n S(x'') \text{ for } x \in U_q(\hat{\mathfrak{g}}),$$

where $S : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$ is the antipode and $\Delta(x) = \sum_{(x)} x' \otimes x''$ is the coproduct $\Delta : U_q \to U_q \otimes U_q$. We also need the Clifford algebra generated by elements ψ_i^n acting in the Fock space and transforming under $U_q(\hat{\mathfrak{g}})$ according to the dual adjoint representation (which in fact is equivalent to the adjoint representation).

The Dirac operator

The Dirac operator Q is acting in $H_f \otimes H_b$ where H_f is the q-fermionic Fock space and H_b carries another highest weight representation of $U_q(\hat{\mathfrak{g}})$.

$$Q = \sum \psi_i^n \otimes B_i^{-n} + \frac{1}{3} \sum \psi_i^n A_i^{-n} \otimes 1$$

where B_i^n is another copy of the adjoint module, acting in the space H_b.

The adjoint module

Let R be the universal R-matrix for the algebra U_q. An explicit construction is given in [KT]. Following [DG], we can then define a basis for vectors in a submodule $A \subset U_q$ transforming according to an adjoint representation

$$ad_q(x)v = \sum_{(x)} x' v S(x'')$$

of U_q on itself. A basis is defined as

$$A_i^n = \sum K_{n,i}^{m,\alpha;p,\beta} (\pi_{m,\alpha;p,\beta} \otimes id) A,$$

where $A = (R^T R - 1)/h$, with $e^h = q$ and $R^T = \sigma R \sigma$, where σ permutes the factors in the tensor product $U_q \otimes U_q$. Here $\pi_{m,\alpha;p,\beta}$ are the matrix elements in the defining representation V of U_q.

For example, for $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$ the basis in the defining representation is v_i^n with $n \in \mathbf{Z}$ and $i = -1, 0, 1$ and $\alpha, \beta = \pm$. The numerical coefficients K come from the identification of the basis of the adjoint representation as linear combinations of the basis vectors in $V \otimes V$.
The action of the Serre generators in the defining representation is

$$\begin{aligned} e_1 v_+^n &= f_0 v_+^n = 0, f_1 v_+^n = v_-^n, e_0 v_+^n = v_-^{n+1}, e_0 v_+^n = v_-^{n-1} \\ e_1 v_-^n &= v_+^n, \; f_0 v_-^n = v_+^{n-1}, e_0 v_-^n = 0 = f_1 v_-^n \\ K_1 v_\pm^n &= q^{\pm 1} v_\pm^n = K_0^{-1} v_\pm^n. \end{aligned}$$

Generalized affine Hecke algebra, $U_q(\widehat{\mathfrak{sl}}(2))$

The affine Hecke algebra for $\hat{\mathfrak{g}}$ is defined through the relations [Leclerc] coming from the R-matrix $\check{\mathrm{R}} = \sigma R$ in the tensor product $V^0 \otimes V^0$. The matrix satisfies

$$(\check{\mathrm{R}} - q^{-1})(\check{\mathrm{R}} + q) = 0,$$

since $-q$ and q^{-1} are the only eigenvalues of the invertible matrix $\check{\mathrm{R}}$. Denote by Y_1 the shift operator which sends $v_i^n \otimes v_j^m$ to $v_i^{n+1} \otimes v_j^m$ and by Y_2 the corresponding shift operator acting on the second tensor factor. The matrix $\check{\mathrm{R}}$ acting on V is then defined using the relations

$$\check{\mathrm{R}} Y_1 = Y_2 \check{\mathrm{R}}^{-1}, \ \check{\mathrm{R}} Y_2 = Y_1 \check{\mathrm{R}} + (q - q^{-1}) Y_2.$$

Actually, the second relation follows from the first and the minimal polynomial relation.

Now the braiding relations are given by setting the ideal in the tensor algebra of V generated by the elements

$$(q^{-1} + \check{\mathrm{R}})(V \otimes V)$$

equal to zero. These have in particular the consequence that any $v_i^n v_j^m$ with $n > m$ can be written as a linear combination of vectors $v_k^p v_l^q$ with $p + q = n + m$ and $p \leq q$. In the zero mode space V^0 the meaning of the braiding relations is that they project out the 'symmetric' part of the tensor product $V^0 \otimes V^0$. The 3-dimensional representation is the eigenspace of $\check{\mathrm{R}}$ with eigenvalue q^{-1} and the 1-dimensional component corresponds to the eigenvalue $-q$.

Generalized affine Hecke algebra

To complete the construction of the Dirac operator we need also the generalized Clifford algebra in the coadjoint representation. The algebra is generated by vectors ψ_i^n with $n \in \mathbf{Z}$ and $i = 1, 0, -1$. The defining relations are given by braiding relations and an invariant (nonsymmetric) bilinear form. The braiding relations are defined recursively like in the case of V, V^*, with the difference that since the R-matrix Ř in the adjoint representation has 3 instead of 2 different eigenvalues, which are now $-q^{-2}, q^2, q^{-4}$, with multiplicities $3, 5, 1$ respectively.

Generalized Hecke algebra

The negative eigenvalue corresponds again to a 3-dimensional 'antisymmetric' representation and the positive eigenvalues to a 6-dimensional 'symmetric' representation; the latter contains the 1-dimensional trivial representation.
The Hecke algebra is replaced by a generalized Hecke algebra,

$$
\begin{aligned}
& Y_1 Y_2 = Y_2 Y_1 \\
& (\check{\mathrm{R}} - q^2)(\check{\mathrm{R}} - q^{-4})(\check{\mathrm{R}} + q^{-2}) = 0 \\
& \check{\mathrm{R}} Y_1 = Y_2 \check{\mathrm{R}}^{-1}, \ \check{\mathrm{R}} Y_2 = Y_i \check{\mathrm{R}} + (q^2 - q^{-2}) Y_2
\end{aligned}
$$

where the middle relation is the minimal polynomial of the diagonalizable matrix $\check{\mathrm{R}}$.

q-Clifford algebra

The generalized symmetric tensors correspond to positive eigenvalues of $\check{R}$. In the Clifford algebra symmetrized products are identified as scalars times the unit. That is, we fix a $U_q(\widehat{\mathfrak{sl}}(2))$ invariant bilinear form B and the Clifford algebra is defined as the tensor algebra over V modulo the ideal generated by

$$P(u \otimes v) - 2B(u, v) \cdot 1$$

where P is the projection on positive spectral subspace of $\check{R}$. In the case when V is the adjoint module for $U_q(\widehat{\mathfrak{sl}}(2))$ one can fix B by identifying the first factor V as the dual V^* and using the natural pairing $V^* \otimes V \to \mathbf{C}$. Alternatively, one can view B as the projection onto the 1-dimensional trivial submodule inside of the 'symmetric module'.

The action of $U_q(\widehat{\mathfrak{sl}}(2))$ on Q

In the nondeformed case one has for an infinitesimal gauge transformation $X \in L\mathfrak{g}$

$$[X, Q] = (k + \kappa) \sum (-n)\psi_i^n X_i^{-n} = (k + \kappa) < \psi, dX >$$

and for a family of operators $Q_A = Q + (k + \kappa)\psi_i^n A_i^{-n}$

$$[X, Q_A] = (k + \kappa) < \psi, [A, X] + dX > .$$

In q-deformed case A is to be understood as a vector in the adjoint module extended by $\mathbf{C}c$. Thus

$$x._c v = x.v + \lambda(x)c$$

with λ a linear form on $U_q(\widehat{\mathfrak{sl}}(2))$.