# POINTWISE CONVERGENCE OF MULTIPLE ERGODIC AVERAGES AND STRICTLY ERGODIC MODELS

WEN HUANG, SONG SHAO, AND XIANGDONG YE

Throughout this paper, by a *topological dynamical system* (t.d.s. for short) we mean a pair $(X, T)$, where $X$ is a compact metric space and $T$ is a homeomorphism from $X$ to itself. A *measurable system* (m.p.t. for short) is a quadruple $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $T : X \to X$ is an invertible measure preserving transformation.

We recall some results related to pointwise ergodic averages. The first pointwise ergodic theorem was proved by Birkhoff in 1931. Followed from Furstenberg's beautiful work on the dynamical proof of Szemeradi's theorem in 1977, problems concerning the convergence of multiple ergodic averages (in $L^2$ or pointwisely) attracts a lot of attention.

The convergence of the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\dots f_d(T^{dn}x) \tag{0.1}$$

in $L^2$ norm was established by Host and Kra [14] (see also Ziegler [21]). We note that in their proofs, the characteristic factors play a great role. The convergence of the multiple ergodic average for commuting transformations was obtained by Tao [19] using the finitary ergodic method, see [5, 13] for more traditional ergodic proofs by Austin and Host respectively. Recently, the convergence of multiple ergodic averages for nilpotent group actions was proved by Walsh [20].

The first breakthrough on pointwise convergence of (0.1) for $d > 1$ is due to Bourgain, who showed in [7] that for $d = 2$, for $p, q \in \mathbb{N}$ and for all $f_1, f_2 \in L^\infty$, the limit $\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^p x) f_2(T^q x)$ exists a.e. Before Bourgain's work, Lesigne showed this convergence holds if the system is distal, with $T^p$, $T^q$ and $T^{p-q}$ ergodic [18]. Also in [9, 2], it was shown that the problem of the almost everywhere convergence of (0.1) can be deduced to the case when the m.p.t. has zero entropy. One can also find some results for weakly mixing transformations in [1, 2, 4].

Recently there are some studies on the limiting behavior of the averages along cubes, and please refer to [6, 14, 3, 8] for details. Also in [8], they obtained the following result. For $i = 1, 2, \dots, d$, let $T_i : X \to X$ be m.p.t., $f_i \in L^\infty(\mu)$ be functions, $p_i \in \mathbb{Z}[t]$ be non-constant polynomials such that $p_i - p_j$ is non-constant for $i \neq j$, and $b : \mathbb{N} \to \mathbb{N}$ be a sequence such that $b(N) \to \infty$ and $b(N)/N^{1/d} \to 0$ as $N \to \infty$, where $d$ is the maximum degree of the polynomials $p_i$. Then the averages

$$\frac{1}{Nb(N)} \sum_{1\le m\le N, 1\le n\le b(N)} f_1(T_1^{m+p_1(n)}x)\dots f_d(T_d^{m+p_l(n)}x)$$

converge pointwise as $N \to \infty$.

Let $(X, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. We say that $(\hat{X}, \hat{T})$ is a *topological model* (or just a *model*) for $(X, \mathcal{X}, \mu, T)$ if $(\hat{X}, \hat{T})$ is a t.d.s. and there exists an invariant probability measure $\hat{\mu}$ on the Borel $\sigma$-algebra $\mathcal{B}(\hat{X})$ such that the systems $(X, \mathcal{X}, \mu, T)$ and $(\hat{X}, \mathcal{B}(\hat{X}), \hat{\mu}, \hat{T})$ are measure theoretically isomorphic.

The well-known Jewett-Krieger's theorem [15, 16] states that every ergodic system has a strictly ergodic model. We note that one can add some additional properties to the topological model. For example, in [17] Lehrer showed that the strictly ergodic model can be required to be a topological (strongly) mixing system in addition.

Now let $\hat{\tau}_d = \hat{T} \times \ldots \times \hat{T}$ ($d$ times) and $\hat{\sigma}_d = \hat{T} \times \ldots \times \hat{T}^d$. The group generated by $\hat{\tau}_d$ and $\hat{\sigma}_d$ is denoted $\langle \hat{\tau}_d, \hat{\sigma}_d \rangle$. For any $x \in \hat{X}$, let $N_d(\hat{X}, x) = \overline{\mathcal{O}((x, \ldots, x), \langle \hat{\tau}_d, \hat{\sigma}_d \rangle)}$, the orbit closure of $(x, \ldots, x)$ ($d$ times) under the action of the group $\langle \hat{\tau}_d, \hat{\sigma}_d \rangle$. We remark that if $(\hat{X}, \hat{T})$ is minimal, then all $N_d(\hat{X}, x)$ coincide, which will be denoted by $N_d(\hat{X})$. It was shown by Glasner [12] that if $(\hat{X}, \hat{T})$ is minimal, then $(N_d(\hat{X}), \langle \hat{\tau}_d, \hat{\sigma}_d \rangle)$ is minimal.

In this paper, first we will show the following theorem.

**Theorem A:** *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. and $d \in \mathbb{N}$. Then it has a strictly ergodic model $(\hat{X}, \hat{T})$ such that $(N_d(\hat{X}), \langle \hat{\tau}_d, \hat{\sigma}_d \rangle)$ is strictly ergodic.*

As a consequence, we have:

**Theorem B:** *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. and $d \in \mathbb{N}$. Then for $f_1, \ldots, f_d \in L^\infty(\mu)$ the averages*

$$\frac{1}{N^2} \sum_{n,m \in [0,N-1]} f_1(T^n x) f_2(T^{n+m} x) \ldots f_d(T^{n+(d-1)m} x) \tag{0.2}$$

*converge $\mu$ a.e.*

We remark that similar theorems as Theorems A and B can be established for cubes. Moreover, the convergence in Theorem B can be stated for any tempered Følner sequence $\{F_N\}_{N \geq 1}$ of $\mathbb{Z}^2$ instead of $[0, N-1]^2$.

It is a long open question if the multiple ergodic averages $\frac{1}{N} \sum_{n=0}^{N-1} \Pi_{j=1}^{d} f_j(T^{jn} x)$ converge a.e. Using some results developed when proving Theorem A, we answer the question positively for distal systems. Namely, we have

**Theorem C:** *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic distal system, and $d \in \mathbb{N}$. Then for all $f_1, \ldots, f_d \in L^\infty(\mu)$*

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \ldots f_d(T^{dn} x)$$

*converge $\mu$ a.e..*

Note that Furstenberg's structure theorem [10, 11] states that each ergodic systems is a weakly mixing extension of an ergodic distal system. Thus, by Theorem C the open question is reduced to deal with the weakly mixing extensions.

## References


[1] E. H. El Abdalaoui, *On the pointwise convergence of multiple ergodic averages*, arXiv:1406.2608

[2] I. Assani, *Multiple recurrence and almost sure convergence for weakly mixing dynamical systems*, Israel J. Math. 103 (1998), 111–124.

[3] I. Assani, *Pointwise convergence of ergodic averages along cubes*, J. Analyse Math. **110** (2010), 241–269.

[4] I. Assani, A.E. Multiple recurrence for weakly mixing commuting actions, arXiv:1312.5270.

[5] T. Austin, *On the norm convergence of non-conventional ergodic averages*, Ergod. Th. and Dynam. Sys., **30**(2010), 321–338.

[6] V. Bergelson. *The multifarious Poincare recurrence theorem*, Descriptive set theory and dynamical systems. London Math. Soc. Lecture Note Series 277, Cambridge Univ. Press, Cambridge, (2000), 31–57.

[7] J. Bourgain, *Double recurrence and almost sure convergence*. J. Reine Angew. Math. 404 (1990), 140–161.

[8] Q. Chu and N. Frantzikinakis, *Pointwise convergence for cubic and polynomial ergodic averages of non-commuting transformations*, Ergod. Th. and Dynam. Sys. **32** (2012), 877-897.

[9] J. Derrien and E. Lesigne, *Un théorème ergodique polynomial ponctuel pourles endomorphismes exacts et les K-systèmes*, Ann. Inst. H. Poincaré Probab. Statist. **32** (1996), no. 6, 765–778.

[10] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*. J. Analyse Math. 31 (1977), 204–256.

[11] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981.

[12] E. Glasner, *Topological ergodic decompositions and applications to products of powers of a minimal transformation*, J. Anal. Math., **64** (1994), 241–262.

[13] B. Host, *Ergodic seminorms for commuting transformations and applications*, Studia Math. 195 (2009), no. 1, 31–49.

[14] B. Host and B. Kra, *Nonconventional averages and nilmanifolds*, Ann. of Math., 161 (2005) 398–488.

[15] R.I. Jewett, *The prevalence of uniquely ergodic systems*, J. Math. Mech. 19 1969/1970 717–729.

[16] W. Krieger, *On unique ergodicity*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, pp. 327–346, Univ. California Press, Berkeley, Calif., 1972.

[17] E. Lehrer, *Topological mixing and uniquely ergodic systems*, Israel J. Math. 57 (1987), no. 2, 239–255.

[18] E. Lesigne, *Théorèmes ergodiques ponctuels pour des mesures diagonales, Cas des systèmes distaux*, Ann. Inst. H. Poincare Probab. Statist. 23 (1987), 593–612.

[19] T. Tao, *Norm convergence of multiple ergodic averages for commuting transformations*, Ergodic Theory Dynam. Systems 28 (2008), no. 2, 657–688.

[20] M. Walsh, *Norm convergence of nilpotent ergodic averages*, Ann. of Math., 175 (2012) 1667–1688.

[21] T. Ziegler, *Universal characteristic factors and Furstenberg averages*. J. Amer. Math. Soc. 20 (2007), no. 1, 53–97.



Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China.

*E-mail address*: `wenh@mail.ustc.edu.cn`

*E-mail address*: `songshao@ustc.edu.cn`

*E-mail address*: `yexd@ustc.edu.cn`