Entropy and independence in symbolic dynamics with connections to number theory
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This is an extended abstract of my talk delivered at MPIM Bonn during the Dynamics and Numbers activity. The talk was based on my joint work with Marcin Kulczycki and Jian Li ([5]).

Abstract

We introduce the subordinate shifts and use them to prove in an elementary way that for every nonnegative real number t one can find a shift space with entropy t. Furthermore, we prove that there is a connection between positive entropy and combinatorial independence of a shift space. Positive entropy can be characterized through existence of a large (in terms of asymptotic and Shnirelman densities) set of coordinates along which the highest possible degree of randomness in points from the shift is observed. This is a well-known fact, but our proof is new and yields a little bit stronger conclusion. It turns out that the shift space known as "square-free flow" and its relatives that have been recently extensively studied are all examples of the subordinate shifts.

Results

Let $\mathbb{N}$ denote the set of *positive* integers. We denote the number of elements of a finite set A by $|A|$. Let $\mathcal{A} = \{0, 1, \dots, r-1\}$ for some $\in \mathbb{N}$. The *full $\mathcal{A}$-shift* is denoted by $\mathcal{A}^{\mathbb{N}}$. A *block over $\mathcal{A}$* is a finite sequence of symbols and its *length* is the number of symbols. The set of all blocks over $\mathcal{A}$ (including the empty word ε) is denoted by $\mathcal{A}^*$.

Definition. We say that a block $w = w_1 \dots w_k \in \mathcal{A}^*$ *dominates* a block $v = v_1 \dots v_k \in \mathcal{A}^*$ if $v_i \leq w_i$ for $i = 1, \dots, k$. A *subordinate of $\mathcal{L} \subset \mathcal{A}^*$* is the set $\mathcal{L}^{\leq}$ of all blocks over $\mathcal{A}$ that are dominated by some block in $\mathcal{L}$. The *subordinate shift of $x \in \mathcal{A}^{\mathbb{N}}$* is the shift space $X^{\leq x}$ given by the language $\mathcal{B}_x^{\leq}$, where $\mathcal{B}_x$ is the language of blocks occurring in x.

Subordinate shifts are *hereditary* (this is a notion introduced in [4] and examined in [7]).

Example. Let η be a point in $\{0,1\}^{\mathbb{N}}$ given by $\eta_n = (\mu(n))^2$, where $\mu \colon \mathbb{N} \to \mathbb{N}$ is the famous Möbius function. It can be shown that $S = X^{\leq \eta}$ is the *square-free flow*, a shift space, whose structure is strongly tied to the statistical properties of square-free numbers, see [1, 6, 9, 12].

Definition. We say that a set $J \subset \mathbb{N}$ is an *independence set* for a shift space $X \subset \mathcal{A}^{\mathbb{N}}$ if for every function $\varphi \colon J \to \mathcal{A}$ there is a point $x = \{x_j\}_{j=1}^{\infty} \in X$ such that $x_i = \varphi(i)$ for every $i \in J$.

Remark. $A \subset \mathbb{N}$ is an independence set for a binary subordinate shift X if and only if the characteristic function of A belongs to X.

Probably the most naturally defined "measure" of a set $A \subset \mathbb{N}$ is the *asymptotic density*

$$d(A) = \lim_{n \to \infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n}.$$

A similar notion is the *Shnirelman density*, which gives information about the structure of $A \cap \{1, \dots, n\}$ for every $n \in \mathbb{N}$ and is defined by

$$d_{\mathrm{Sh}}(A) = \inf \left\{ \frac{|A \cap \{1, 2, \dots, n\}|}{n} : n \in \mathbb{N} \right\}.$$

Our main result:

Theorem 1. *Let X be a binary shift. Then the entropy of X is positive if and only if X is independent over a set A whose asymptotic density exists, is positive, and is equal to its Shnirelman density.*

This is a strengthening of [13, Theorem 8.1]. See also [2], [3, Thm. 7.3], or [4]. We add the Shnirelman density to the picture, which shows that the independence set is even more structured.

Below are the main tools of our proof. Fix a shift space X over $\mathcal{A}$ and let $a \in \mathcal{A}$ and $x \in X$.

Definition. We define $\chi_a(x) = \{j \in \mathbb{N} : x_j = a\}$. Let $||w||_a$ denote the number of a's in w. Let $\mathrm{M}_k^a(X)$ be the maximal number of occurrences of the symbol a among all blocks $w \in \mathcal{B}_k(X)$. The sequence $\{\mathrm{M}_k^a(X)\}_{k=1}^\infty$ is non-negative and subadditive, hence the sequence $\{\mathrm{M}_k^a(X)/k\}_{k=1}^\infty$ converges to its greatest lower bound, which we denote $\mathrm{Fr}_a(X)$.

The next theorem is a strengthening of a well-known result about the ordinary density. It also follows from [10, Theorem 4, p. 323] (see also [8]).

Theorem 2. *There exists a point $\omega_a \in X$ such that $d_{Sh}(\chi_a(\omega_a)) = d(\chi_a(\omega_a)) = \mathrm{Fr}_a(X)$.*

Definition. Let $\mathcal{F}$ be a (possibly empty) family of binary blocks of length $n \geq 0$. We say that $\mathcal{F}$ is *independent over a set* $J \subset \mathbb{N}$ and J is an *independence set for* $\mathcal{F}$ if for each map $\varphi\colon J \to \{0,1\}$ there is a block $w \in \mathcal{F}$ whose i-th symbol is $\varphi(i)$ for every $i \in J$. We denote the collection of all sets of independence for $\mathcal{F}$ by $\mathcal{I}(\mathcal{F})$.

Lemma 1. *Let $\mathcal{F}$ be a family of binary blocks of length $n \geq 0$. Then $|\mathcal{I}(\mathcal{F})| \geq |\mathcal{F}|$.*

Lemma 2. *Let $\mathcal{F} \subset \{0,1\}^n$ be a family of binary blocks of length $n \geq 1$. If for some $1 \leq k \leq n$ we have $|\mathcal{F}| > \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k-1}$, then $\mathcal{F}$ is independent over some set of cardinality k.*

Lemma 3. *Let X be a binary shift with positive topological entropy. Then there is an $\varepsilon > 0$ such that for every $n \geq 1$ there is a set $J \subset \{1, \ldots, n\}$ with $\lfloor \varepsilon n \rfloor$ elements which is an independence set for X.*

References

[1] El Abdalaoui et al., *A dynamical point of view on the set of $\mathcal{B}$-free integers.* Preprint (2013) `arXiv:1311.3752[math.DS]`

[2] E. Glasner, B. Weiss, *Quasi-factors of zero-entropy systems*, J. Amer. Math. Soc. 8 (1995), no. 3, 665–686.

[3] W. Huang, X. Ye, *A local variational relation and applications*, Israel J. Math. 151 (2006), 237–279.

[4] D. Kerr, H. Li, *Independence in topological and C^*-dynamics.* Math. Ann. 338 (2007), no. 4, 869–926.

[5] M. Kulczycki, D. Kwietniak, J. Li, *Entropy and independence in symbolic dynamics*, `arXiv:1401.5969v1[math.DS]`, preprint 2014.

[6] J. Kułaga-Przymus, Joanna, M. Lemańczyk, B. Weiss, *On invariant measures for $\mathcal{B}$-free systems.* Preprint (2014) `arXiv:1406.3745[math.DS]`

[7] D. Kwietniak, *Topological entropy and distributional chaos in hereditary shifts with applications to spacing shifts and beta shifts.* Discrete Contin. Dyn. Syst. 33 (2013), no. 6, 2451–2467.

[8] Y. Peres, *A combinatorial application of the maximal ergodic theorem.* Bull. London Math. Soc. 20 (1988), no. 3, 248–252.

[9] R. Peckner, *Uniqueness of the measure of maximal entropy for the squarefree flow.* Preprint (2014) `arXiv:1205.2905v6[math.DS]`

[10] I. Z. Ruzsa, *On difference sets.* Studia Sci. Math. Hungar. 13 (1978), no. 3-4, 319–326 (1981).

[11] Sauer, N. On the density of families of sets. J. Combinatorial Theory Ser. A 13 (1972), 145–147.

[12] P. Sarnak, *Three lectures on the Möbius function, randomness, and dynamics.* Online notes, `http://publications.ias.edu/sarnak/paper/512` (2011).

[13] B. Weiss, Single orbit dynamics. CBMS Regional Conference Series in Mathematics, 95. American Mathematical Society, Providence, RI, 2000.