Mean equicontinuity and mean sensitivity

Jian Li

(joint work with Siming Tu and Xiangdong Ye)

Department of Mathematics, Shantou University, Shantou,
Guangdong 515063, P.R. China

lijian09@mail.ustc.edu.cn

Abstract: In this talk, we study equicontinuity and sensitivity in the mean sense. We show that every ergodic invariant measure of a mean equicontinuous (i.e. mean-L-stable) system has discrete spectrum. Dichotomy results related to mean equicontinuity and mean sensitivity are obtained when a dynamical system is transitive or minimal. Localizing the notion of mean equicontinuity, notions of almost mean equicontinuity and almost Banach mean equicontinuity are introduced. It turns out that a system with the former property may have positive entropy and meanwhile a system with the later property must have zero entropy.

By a *topological dynamical system* (X,T) we mean a compact metric space X with a continuous map T from X into itself; the metric on X is denoted by d.

A dynamical system (X,T) is called *equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x,y \in X$ with $d(x,y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for $n = 0,1,2,\dots$, that is, the family of maps $\{T^n \colon n \in \mathbb{Z}_+\}$ is uniformly equicontinuous. Equicontinuous systems have simple dynamical behaviors. It is well known that a dynamical system (X,T) with T being surjective is equicontinuous if and only if there exists a compatible metric ρ on X such that T acts on X as an isometry, i.e., $\rho(Tx,Ty) = \rho(x,y)$ for any $x,y \in X$. Moreover, a transitive equicontinuous system is conjugate to a minimal rotation on a compact abelian metric group, and (X,T,μ) has discrete spectrum, where μ is the unique Haar measure on X.

When studying dynamical systems with discrete spectrum, Fomin [8] introduced a notion called *stable in the mean in the sense of Lyapunov* or simply *mean-L-stable*. A dynamical system (X,T) is *mean-L-stable* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ except a set of upper density less than ε. Fomin proved that if a minimal system is mean-L-stable then it is uniquely ergodic. Mean-L-stable systems are also discussed briefly by Oxtoby in [26], and he proved that each transitive mean-L-stable system is uniquely ergodic. Auslander in [2] systematically studied mean-L-stable systems, and provided new examples. See Scarpellini [27] for a related work. It is an open questions if every ergodic invariant measure on a mean-L-stable system has discrete spectrum [27]. We will give an affirmative answer to this question.

We introduce equicontinuity in the mean sense, more precisely, a dynamical system (X,T) is called *mean equicontinuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x,y \in X$ with $d(x,y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

We show that a dynamical system is mean equicontinuous if and only if it is mean-L-stable. We also show each dynamical system admits a maximal mean equicontinuous factor, and mean equicontinuity is preserved by factor maps. We remark that studying dynamical properties in the mean sense is an interesting topic, see [25] for the research on mean distality and [7, 19] for the investigation on mean Li-Yorke chaos.

The notion of sensitivity was introduced when studying the complexity of a dynamical system, and it is a part of the known definition of chaos in the Devaney sense. We say that a dynamical system (X,T) has *sensitive dependence on initial condition* or briefly (X,T)

is *sensitive* if there exists a $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ and $n \in \mathbb{N}$ such that $d(T^n x, T^n y) > \delta$.

When considering the opposite side of sensitivity the notion of equicontinuity at a point appears naturally, see [15]. That is a point $x \in X$ is called an *equicontinuous point* (or (X,T) is *equicontinuous at* x) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $y \in X$ with $d(x,y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$. If every point in X is an equicontinuous point then by the compactness of X the dynamical system (X,T) is equicontinuous. A transitive system is called *almost equicontinuous* if there is at least one equicontinuous point. Almost equicontinuous systems have been studied intensively and have many applications. For example, the enveloping semigroup $E(X)$ is metrizable if and only if (X,T) is hereditarily almost equicontinuous [14].

We know that if (X,T) is almost equicontinuous then the set of equicontinuous points coincides with the set of all transitive points [1], it is uniformly rigid [15] and thus has zero topological entropy [13]. We have the following dichotomy results. If (X,T) is minimal, then (X,T) is either equicontinuous or sensitive [4]; and if (X,T) is transitive, then (X,T) is either almost equicontinuous or sensitive [1].

Inspirited by the above ideas, we will introduce notions of almost mean equicontinuity and mean sensitivity. A point $x \in X$ is called *mean equicontinuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $y \in X$ with $d(x,y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

A transitive system is called *almost mean equicontinuous* if there is at least one mean equicontinuous point. A dynamical system (X,T) is called *mean sensitive* there exists a $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ and $n \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

We show that if a dynamical system (X,T) is minimal, then (X,T) is either mean equicontinuous or mean sensitive, and if (X,T) is transitive, then (X,T) is either almost mean equicontinuous or mean sensitive. Unlike the case of almost equicontinuous systems, we show that for almost mean equicontinuous systems the set of transitive points is contained in the set of all mean equicontinuous points and there are examples in which they do not coincide. It is unexpected that there are almost mean equicontinuous systems admitting positive topological entropy, while every almost equicontinuous system has zero topological entropy.

Thus it is natural to seek a class of mean equicontinuous systems for which the localized systems at least have zero entropy. We find the class of *Banach mean equicontinuous systems* obtained by replacing small upper density with small Banach density in the definition of mean-L-stable systems is the right one. Namely we show that almost Banach mean equicontinuous systems have zero topological entropy, and this implies that the almost mean equicontinuous systems admitting positive topological entropy are not almost Banach mean equicontinuous. The deep reason of this is that in a transitive system a transitive point can only approach a "chaotic subsystem" for time segments, which may result large Banach density and at the same time small upper density.

References

[1] E. Akin, J. Auslander and K. Berg, *When is a transitive map chaotic?*, Convergence in ergodic theory and probability (Columbus, OH, 1993), 25–40, Ohio State Univ. Math. Res. Inst. Publ., 5, de Gruyter, Berlin, 1996.

[2] J. Auslander, *Mean-L-stable systems*, Illinois J. Math., **3** (1959), 566–579.

[3] J. Auslander, *Minimal Flows and Their Extensions*, North-Holland Publishing Co., Amsterdam, 1988.

[4] J. Auslander and J. A. Yorke, *Interval maps, factors of maps, and chaos*, Tohoku Math. J., **32** (1980), no. 2, 177–188.
[5] F. Blanchard, *A disjointness theorem involving topological entropy*, Bull. de la Soc. Math. de France, **121**(1993), 465–478.
[6] R. Ellis and W. H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Amer. Math. Soc. **94** (1960), 258–271.
[7] T. Downarowicz, *Positive topological entropy implies chaos DC2*, Proc. Amer. Math. Soc., **142** (2014), 137–149.
[8] S. Fomin, *On dynamical systems with a purely point spectrum*, Dokl. Akad. Nauk SSSR, vol. **77** (1951), 29–32 (In Russian).
[9] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton, NJ, 1981.
[10] F. García-Ramos, *Weak forms of topological and measure theoretical equicontinuity: relationships with discrete spectrum and sequence entropy*, arXiv:1402.7327v2[math.DS].
[11] J. Gillis, *Notes on a property of measurable sets*, J. London Math. Soc., **11** (1936), 139–141.
[12] E. Glasner, *The structure of tame minimal dynamical systems*, Ergod. Th. & Dynam. Sys., **27** (2007), 1819–1837.
[13] S. Glasner and D. Maon, *Rigidity in topological dynamics*, Ergod. Th. & Dynam. Sys., **9** (1989), 309–320.
[14] E. Glasner, M. Megrelishvili and V. Uspenskij, *On metrizable enveloping semigroups*. Israel J. Math., **164** (2008), 317–332.
[15] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity, **6** (1993), no. 6, 1067–1075.
[16] F. Hahn, Y. Katznelson, *On the entropy of uniquely ergodic transformations*, Trans. Amer. Math. Soc., **126** (1967), 335–360.
[17] W. Huang, S. Li, S. Shao and X. Ye, *Null systems and sequence entropy pairs*, Ergod. Th. & Dynam. Sys., **23** (2003), 1505–1523.
[18] W. Huang, P. Lu and X. Ye, *Measure-theoretical sensitivity and equicontinuity*, Israel J. of Math., **183** (2011), 233–283.
[19] W. Huang, J. Li and X. Ye, *Stable sets and mean Li-Yore chaos in positive entropy systems*, Journal of Functional Analysis, **266** (2014), 3377–3394.
[20] W. Huang and X. Ye, *Devaneys chaos or 2-scattering implies Li-Yorkes chaos*, Topol. Appl., **117** (2002), 259–272.
[21] W. Huang and X. Ye, *Minimal sets in almost equicontinuous systems*, Proc. of the Steklov Inst. of Math., **244** (2004), 280–287.
[22] W. Huang and X. Ye, *A local variational relation and applications*, Israel J. Math., **151** (2006) 237–280.
[23] D. Kerr and H. Li, *Independence in topological and C^*-dynamics*, Math. Ann., **338** (2007), 869–926.
[24] J. Li and S. Tu, *On proximality with Banach density one*, J. Math.Anal.Appl. **416** (2014), 36–51.
[25] D. Ornstein and B. Weiss, *Mean distality and tightness*, Proc. Steklov Inst. Math., **244** (2004), no.1, 295–302.
[26] J. C. Oxtoby, *Ergodic sets*, Bull. Amer. Math. Soc., **58** (1952), 116–136.
[27] B. Scarpellini, *Stability properties of flows with pure point spectrum*, J. London Math. Soc. (2) **26** (1982), no. 3, 451–464.
[28] P. Walters, *An introduction to ergodic theory*. Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.
[29] X. Ye and R. Zhang, *On sensitivity sets in topological dynamics*, Nonlinearity, **21** (2008), 1601–1620.